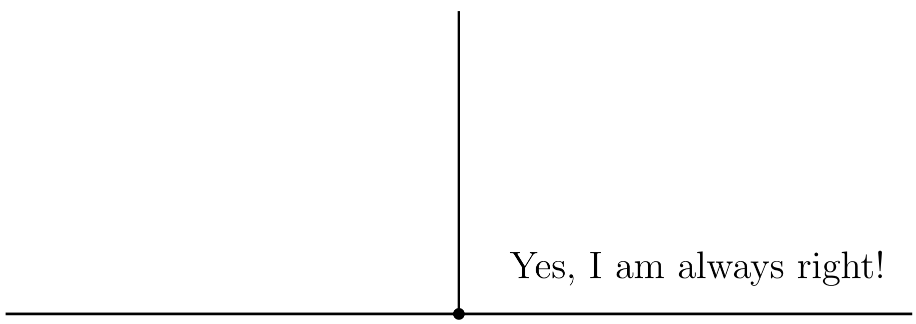
Oleg Gleizer LAMC

Introduction to Geometry Lesson 3

The right angle

An angle is called *right* if it is congruent to its supplementary angle.



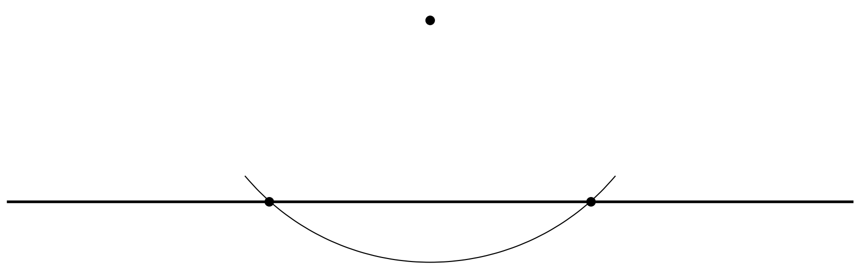
An angle smaller than a right angle is called *acute*. An angle larger than a right angle, but smaller than a straight angle is called *obtuse*.

Problem 1 *Use degrees to write an algebraic statement showing that the angle α is*

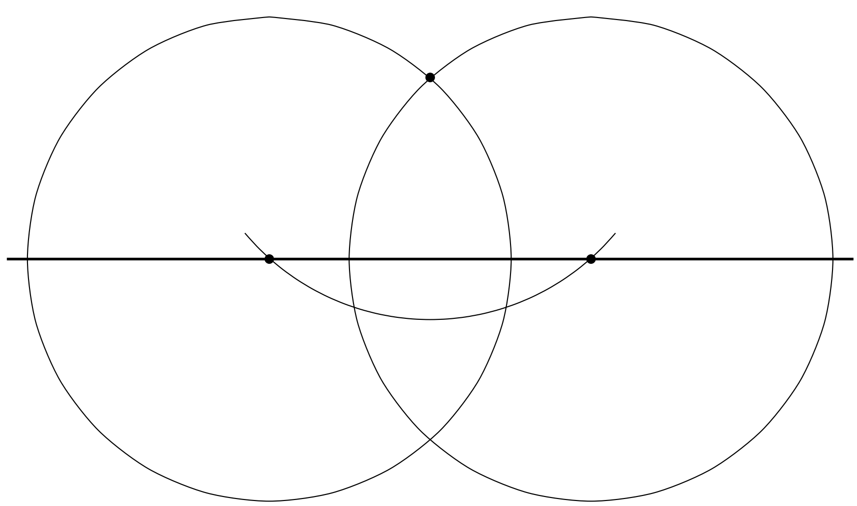
- *right:*
- *acute:*
- *obtuse:*

Given a straight line and a point not lying on the line, the following procedure enables one to construct the right angle such that one of its sides is a part of the line and the other passes through the point, using a compass and straightedge as tools.

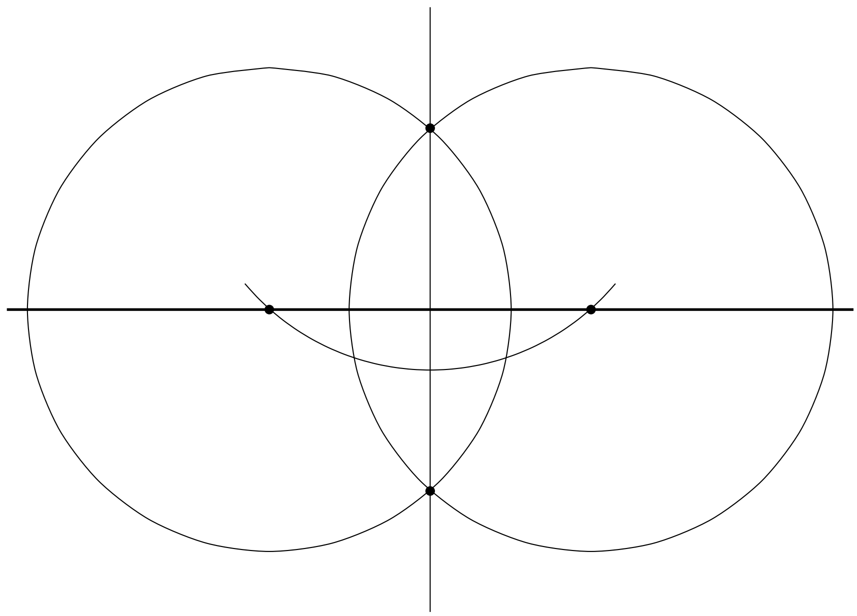
Step 1: spread the legs of the compass wide enough so that the circumference centered at the given point meets the line at two distinct points. Mark the points.



Step 2: keeping the radius the same, draw the circumferences centered at the two points on the line.

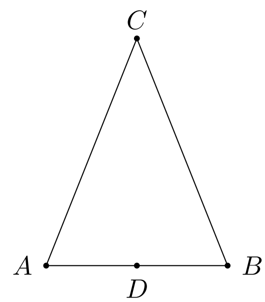


Step 3: mark the point opposite to the original one and draw a straight line through the two points.



The following sequence of problems explains why the above method works and explores some of the features of the involved geometric objects.

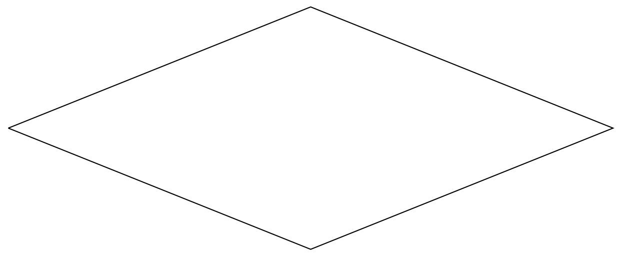
Recall that a triangle is called *isosceles* if it has two sides of equal length. Consider the triangle ABC such that $AC = BC$. Let D be the midpoint of the side AB.



Problem 2 *Prove that the angle ADC is right.*

Problem 3 *Prove that the angles of an isosceles triangle opposite to the congruent sides are congruent.*

A quadrilateral is called a *rhombus* if all its sides have the same length.



Problem 4 *Prove that opposite angles of a rhombus are congruent.*

Problem 5 *Prove that diagonals split angles of a rhombus in halves.*

Problem 6 *Prove that diagonals of a rhombus intersect at a right angle.*

Question 1 *Does Problem 6 explain why the method of constructing a right angle presented at the beginning of the lesson works? Why or why not?*

Problem 7 *Prove that diagonals of a rhombus split each other in halves.*

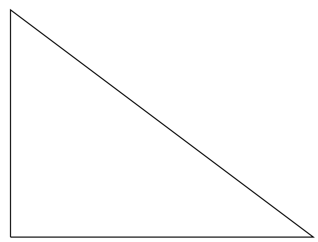
Problem 8 *Use a compass and a straightedge to find the midpoint of the following segment.*

A quadrilateral with all the four sides of equal length and all the four angles right is called a *square*.

Problem 9 *Use a compass and a ruler to construct a square with* $2''$ *side lengths in the space below.*

The Pythagoras Theorem

A triangle is called *right*, if it has a right angle.

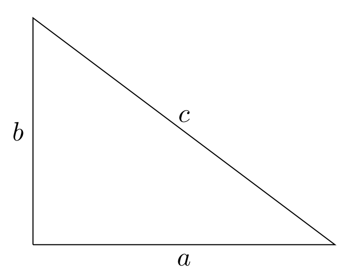


We will later prove that in the Euclidean (a.k.a. flat) plane, a triangle can have only one right angle. This is not necessarily the case on curvy surfaces.

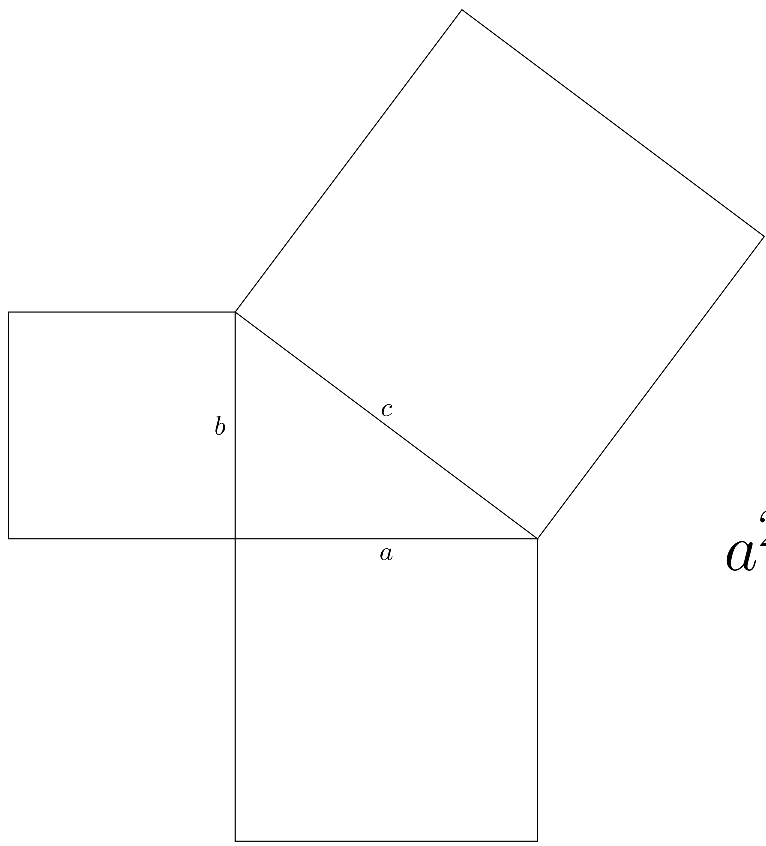
Homework Problem 1 *On the globe, find a triangle formed by the equator and two meridians that has three right angles.*

The sides of a right triangle forming the right angle are called its *legs*, or *catheti* (singular *cathetus*). The side opposite to the right angle is called the *hypotenuse*.

Consider a right triangle with the legs' lengths a and b and with the hypotenuse of length c.



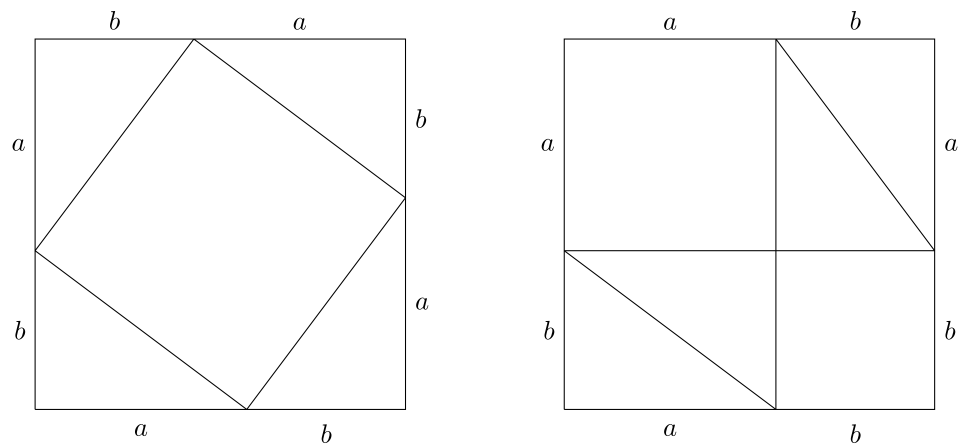
Theorem 1 (Pythagoras) *The sum of the areas of the squares built on the legs of a right triangle is equal to the area of the square built on its hypotenuse.*



$$a^2 + b^2 = c^2$$

A philosophical note: the theorem was known to Egyptian priests about 2,000 years before Pythagoras was born. As observed by a great Russian mathematician Vladimir Arnold, a mathematical discovery is most often named after the last person to make it.

Problem 10 *Prove the Pythagoras' theorem by comparing the two tilings of a square with a side length $a + b$ below.*



The following statement is inverse to the Pythagoras theorem.

Theorem 2 *If the area of the square built on a side of a triangle (in the Euclidean plane) equals to the sum of the areas of the squares built on the other two sides of the triangle, then the angle opposite to the first side is right.*

Problem 11 *Prove Theorem 2. Hint: use the Pythagoras' theorem and the SSS congruence.*

Theorems 1 and 2 are often combined under the name of the Pythagoras' theorem.

Theorem 3 *A triangle (in the Euclidean plane) is right if and only if the area of the square built on one of its sides equals to the sum of the areas of the squares built on the other two sides.*

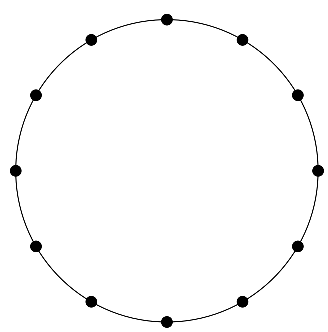
The numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ are called *natural*.

A triple of natural numbers a, b, and c is called a *Pythagorean triple*, if $a^2 + b^2 = c^2$.

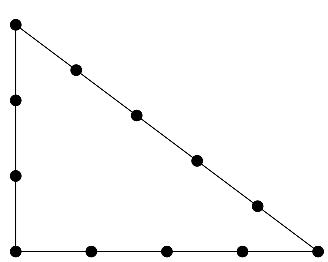
Problem 12 *Check if the numbers* 3, 4, *and* 5 *form a Pythagorean triple.*

A very practical application: the theorem later attributed to Pythagoras was invented as a construction tool. Imagine yourself an Egyptian priest tasked with making sure that two adjacent pyramid walls meet at the right angle.

Take a rope and tie it into a circle divided into 12 pieces by equally spaced knots.



Then stretch the rope into a $(3, 4, 5)$ triangle



and, thanks to Theorem 2, you get the right angle!

Gizah pyramids

Note 1 *Note that Theorem 2 is more practical than Theorem 1. However, we need Theorem 1 to prove Theorem 2.*

The $(3, 4, 5)$ rope is still occasionally used to delineate a soccer or cricket field.

Problem 13 *Find a Pythagorean triple different from* $(3, 4, 5)$.

The area of a square of side length $a+b$ is $(a+b)\times(a+b) = (a+b)^2$.

Problem 14 *Use a picture similar to the one utilized in proving Theorem 1 to prove the following algebraic identity.*

$$(a+b)^2 = a^2 + 2ab + b^2 \tag{1}$$

Problem 15 *Use formula (1) to find* 20.3^2 *without a calculator.*

Problem 16 *[1] Without using a calculator, find the Pythagorean triple with the hypotenuse's length of 73 cm.*

Two straight lines are called *perpendicular* or *orthogonal* if they form a right angle.

Problem 17 *Use a compass and a straightedge to construct a straight line perpendicular to the given one and passing through the given point on the line.*

Self-test questions

- What is a right angle?

- What is the measure of a right angle in degrees?

- What is an acute angle?

- What is an obtuse angle?

- Why are the angles of an isosceles triangle opposite to the congruent sides congruent?

- What is a rhombus?

- Why do the diagonals of a rhombus intersect at the right angle and split each other in halves?

- How to construct a straight line perpendicular to a given one and passing through a given point outside of the original line? On the original line?

- Formulate the Pythagoras' theorem. How do they prove it?

- Formulate the inverse of the Pythagoras' theorem. How do they prove this one?

- What is a Pythagorean triple?

- How can one use a $(3, 4, 5)$-rope in practice?

References

[1] *Introduction to Geometry*, R. Rusczyk, Art of Problem Solving, ISBN 978-1-934124-08-6