The Boolean Algebra and Logic Gates

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Adapted from a series of worksheets by Oleg Gleizer

1 The Algebra of Statements, Continued

Recall from last week that we learned how to add two statements A and B to form the statement $A + B$. Similar to the logical addition, we can introduce logical multiplication. Let us define $A \times B$ as the statement *A and B*. For example, if $A =$ *three is greater than two* and $B =$ *three is greater than five*, then $A \times B =$ *three is greater than two and five*. Quite obviously, *A and B* is true if and only if both A and B are true.

A	B	$A \times B$
0	0	0
1	0	0
0	1	0
1	1	1

In the algebra of logic just like in the algebra of numbers, $0 \times 0 = 1 \times 0 = 0 \times 1 = 0$, while $1 \times 1 = 1$.

Problem 1 *Is the logical multiplication commutative? Why or why not? Please write down your explanation in the space below.*

Problem 2 *Prove that $A \times 0 = 0$ and $A \times 1 = A$.*

Problem 3 *Prove that* $\underbrace{A \times A \times \ldots \times A}_{n\ times} = A$

Problem 4 *Prove that for the logical multiplication,*
$(A \times B) \times C = A \times (B \times C)$.

A	B	C	$A \times B$	$B \times C$	$(A \times B) \times C$	$A \times (B \times C)$
0	0	0				
1	0	0				
0	1	0				
0	0	1				
1	1	0				
1	0	1				
0	1	1				
1	1	1				

Problem 5 *Form the logical product of the following three statements and find its value.*

$A =$ *Lobsters live in the ocean.*

$B =$ *Mobsters do not live in the ocean.*

$C =$ *Lobsters are no better than mobsters.*

$A \times B \times C =$

Problem 6

$A =$ *I start to like the science of logic.*

$B =$ *"To Kill a Mockingbird" is a hunters' guidebook.*

$C = 48$ *is divisible by* 12.

Form the following statements from the above A, B, *and* C *and find their value.*

$A \times (B + C) =$

$A \times B + A \times C =$

Problem 7 *Prove that*
$A \times (B + C) = (B + C) \times A = A \times B + A \times C.$

A statement A preceded by *it is not true that ...* or a statement equivalent to such is called the *negation* of A and is denoted as $\neg A$. For example, the negation of the statement B from Problem 6 reads as follows. *"To Kill a Mockingbird" is not a hunters' guidebook.* The following is the truth table for the negation.

A	$\neg A$
0	1
1	0

Problem 8 *Write down the negation of the statement "I come to the Math Circle by car or by bus" in the space below.*

Problem 9 *Write down your own composite statement and its negation.*

Statement:

Negation:

Problem 10 *Can the statement $A + \neg A$ be false? Why or why not?*

Write the fact down as an algebraic formula.

Problem 11 *Can the statement $A \times \neg A$ be true? Why or why not?*

Write the fact down as an algebraic formula.

Problem 12 *Find the following.*

$\neg\neg A =$

Problem 13 *Given the statements*

A: Bob is driving to work.

B: Bob is shaving.

C: Bob is eating a burger.

form the following statements.

- $AB + \neg C =$

- $(A + B)C =$

- $A\neg B + C =$

- $\neg A \neg BC =$

Problem 14 *Using the simple statements A, B, and C from Problem 13, rewrite the following as a mathematical formula.*

It is not true that Bob is either driving to work and shaving or eating a burger.

The algebra of logic we have studied above is called *Boolean*, after George Boole (1815-1864), an English mathematician, philosopher and logician.

Below you will find one more feature of the algebra that truly distinguishes it from every other algebraic structure you have seen before.

In Problem 7, we have proven that, similar to the algebra of numbers, multiplication in Boolean algebra is distributive.

$$A \times (B + C) = A \times B + A \times C$$

In Boolean algebra, unlike the algebra of numbers, addition is distributive with respect to multiplication as well!

$$A + (B \times C) = (A + B) \times (A + C) \tag{1}$$

Problem 15 *Prove formula 1.*

Problem 16 *Find the value of the composite statement $A + B\neg C$ for the simple statements A, B, and C below.*

A: California is a part of kangaroo's natural habitat.

B: Math Kangaroo is at home at UCLA.

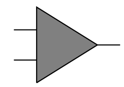
C: Statements A and B contradict each other.

$A + B\neg C =$

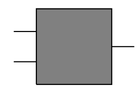
2 Logic Gates

Logic gates are hardware realizations of the Boolean algebra operations that power our computers and smartphones. We draw them as in the picture below.

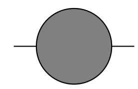
Or gate



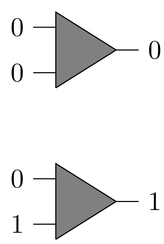
And gate

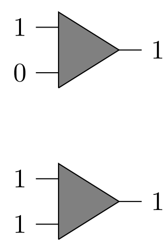


Not gate



The **Or** gate operates as follows. If at least one of the inputs has current, denoted as 1 on the picture below, then the output has current as well. If none on the inputs has current, absence of the current denoted by 0, then neither has the output.





As you can see, the **Or** gate is simply a hardware implementation of the truth table for the logical addition.

A	B	$A+B$
0	0	0
1	0	1
0	1	1
1	1	1

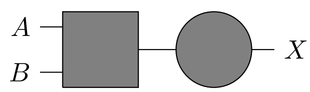
Problem 17 *Fill out the truth table for the logical multiplication* **And**.

A	B	$A \times B$
0	0	
1	0	
0	1	
1	1	

Problem 18 *Mark the inputs and outputs of the* **And** *logic gate corresponding to the truth table from Problem 17.*

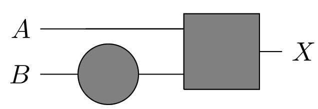
The **Not** gate, also known as an *inverter*, always inverts the incoming signal.

Problem 19 *What is the value of the output X for the input values $A = 1$ and $B = 0$?*



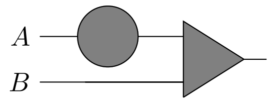
$X =$

Problem 20 *Find the value of the output X for the following input values.*



- $A = 0,\ B = 1, \quad X =$
- $A = 1,\ B = 1, \quad X =$

Problem 21 *Find the value of the output X for the input values $A = 0$ and $B = 0$.*



$X =$

The following theorems are called *De Morgan's laws.*

$$\neg(A + B) = \neg A \times \neg B \qquad\qquad \neg(A \times B) = \neg A + \neg B$$

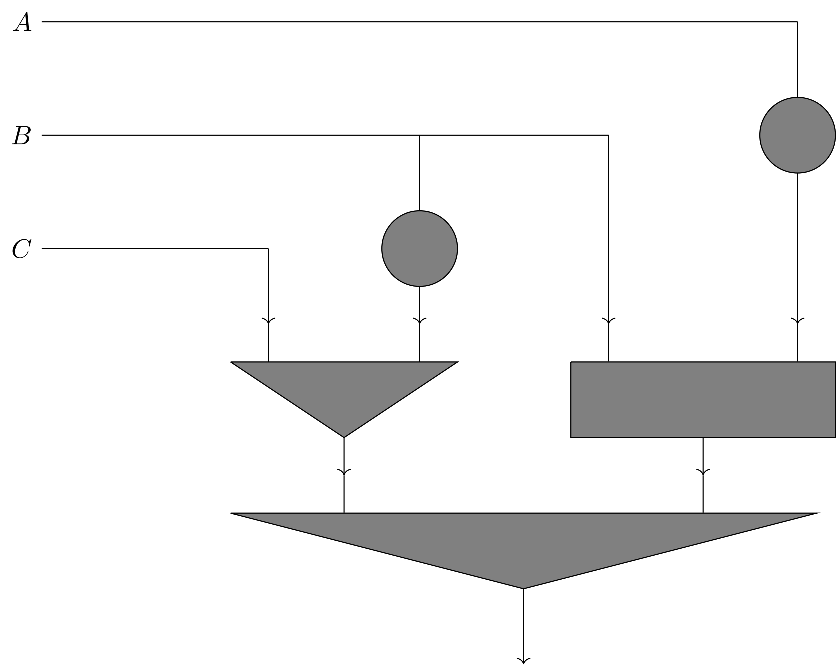
Problem 22 *Use logic gates circuits to prove the first of De Morgan's laws.*

Problem 23 *Use logic gates circuits to prove the second of De Morgan's laws,* $\neg(A \times B) = \neg A + \neg B$.

Problem 24 *Realize the* **And** *gate by means of a circuit that has inverters and* **Or** *gates only.*

Problem 25 *Realize the* **Or** *gate by means of a circuit that has inverters and* **And** *gates only.*

Problem 26 *What Boolean algebra formula is implemented by the following circuit?*

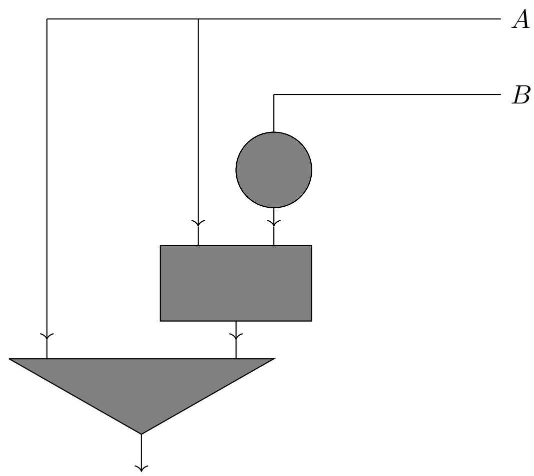


Problem 27 *Draw the logic gates circuit that implements the following Boolean algebra formula.*

$$\neg AB\neg(A+\neg B)$$

Hint: it helps to simplify the formula first.

Problem 28 *Simplify the following circuit.*



Problem 29 *In a two-story townhouse, there is one electric light over the stairs from the first to the second floor. The light has two switches, switch A at the bottom and switch B at the top of the stairs. Design the circuit satisfying the following requirements.*

- *When someone at the bottom of the stairs turns A on, the light is on. B is assumed to be in the off position.*

- *With A on, switching B to the on position turns off the light.*

- *If a new person enters the house and flips A to the off position, with B still in the on mode, the light turns on.*

- *With A in the off position, the person upstairs can turn off the light by switching B to the off mode.*

Problem 30 *Design a circuit for adding two one-digit binary numbers.*

Problem 31 *Design a circuit for adding three one-digit binary numbers.*