ORMC Olympiad Group
Week 2 Solutions

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Solutions

Solution 1

(a) Let a_n be the number of ways for a $1 \times n$ rectangle. Consider the rightmost domino. If it is a 1×1 domino, then the remaining rectangle is $1 \times (n-1)$, so there are a_{n-1} ways to finish the covering in this case. Similarly, the rightmost domino being 1×2 or 1×3 leads to a_{n-2} and a_{n-3} ways to finish, respectively. So,

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}.$$

We also have the base cases of $a_0 = 1, a_1 = 1, a_2 = 2$, so we can compute up to $a_{10} = \boxed{274}$.

(b) Let a_n be the number of ways for a $1 \times n$ rectangle. Similarly to above, we consider cases on the rightmost domino to get the recurrence relation of

$$a_n = a_{n-1} + a_{n-3}.$$

The bases cases are now $a_0 = 1, a_1 = 1, a_2 = 1$, so $a_{10} = \boxed{28}$.

Solution 2

Let a_n be the number of ways to cover a $2 \times n$ rectangle, and consider the height of the rectangle to be 2. Again, consider the rightmost dominos.

There are three cases: if it is a 2×2 domino, then we now have to cover a $2 \times (n-2)$ rectangle. If it is a 1×2 domino with the height as 2, then we now have a $2 \times (n-1)$ rectangle. Also, the third case is that there are two 1×2 rectangles with the height as 1, stacked on top of each other. In this case there is also a $2 \times (n-2)$ rectangle remaining. So, the recurrence is

$$a_n = a_{n-1} + 2a_{n-2}.$$

The base cases are $a_1 = 1$ and $a_2 = 3$. So we can compute $a_{10} = \boxed{683}$.

Solution 3

There are two states. We will refer to them as a, when the ant is at A, and b, when the ant is not at A. Then $p_a(t)$ will refer to the probability the ant is at vertex a after t steps, and similarly for $p_b(t)$. Note that $p_a(0) = 1$ and $p_b(0) = 0$.

If the ant is at A, it will move to a different vertex with probability 1 in the next step. If it is not at A, there is an equal chance that it will go to A or stay on not A. So,

$$p_a(t) = \frac{1}{2}p_b(t-1), \qquad p_b(t) = p_a(t) + \frac{1}{2}p_b(t-1).$$

Now we just compute the values using the base case and find $p_a(6) = \boxed{\frac{11}{32}}$.

Solution 4

If the ant starts at vertex A, let $s_a(t)$ represent the number of ways for the ant to go to vertex A in t steps, and let $s_b(t)$ be the same except going to vertex B, and define $s_c(t)$ similarly. Note that $s_b(t) = s_c(t)$ for all t. We want to find $s_a(8)$.

Note the base case of $s_a(0) = 1$ and $s_b(0) = s_c(0) = 0$. Now consider $s_a(n)$. If the ant is at A after n steps, in the previous step it must have been at either B or C. So $s_a(n) = s_b(n-1) + s_c(n-1)$. Similarly, we also have $s_b(n) = s_a(n-1) + s_c(n-1)$ and $s_c(n) = s_a(n-1) + s_b(n-1)$.

Now we just use the base cases to compute from $t = 0$ to 8, and find $s_a(8) = \boxed{86}$.

Solution 5

Let a_n represent the amount of ways for the frog to go up n stairs. Note that $a_1 = 0, a_2 = 1, a_3 = 1$.

To go up a flight of n stairs, first the frog must jump either 2 or 3 stairs up. This leaves two cases: either $n-2$ or $n-3$ stairs left. So, the recurrence is $a_n = a_{n-2} + a_{n-3}$. We can then calculate a_n for $1 \leq n \leq 19$ using the base cases above, giving $a_{19} = \boxed{86}$.

Solution 6

Let a_n be the number of ways for n houses. The constraints are that no two houses in a row both get mail, and no three houses in a row all do not get mail. Now we consider the last house.

If the last house receives mail, then the second to last must not receive mail. Now it seems that we must casework on the third to last house. But if we instead consider the houses from $1, \ldots, n-3$, simply with the original constraints of the problem, there are a_{n-3} ways to do this. Then, note that whether or not the $(n-2)$-th house gets mail is now fixed: if house $n-3$ got mail, it clearly cannot also; but if $n-3$ did not get mail, since $n-1$ also did not get mail, then $n-2$ must get mail. Thus there are simply a_{n-3} delivery patterns in this case.

If the last house does not receive mail, then similarly, ignore house $n-1$ for now and just fill in the delivery pattern for the first $n-2$ houses. There are a_{n-2} ways to do so. Then, just like above, the delivery status of house $n-1$ is now fixed, so there are simply a_{n-2} delivery patterns in this case.

So, the recursion is $a_n = a_{n-2} + a_{n-3}$. The base cases are $a_1 = 2, a_2 = 3$, and $a_3 = 4$, so then we can compute up to $a_{19} = \boxed{351}$.

Solution 7

Let a_n be the number of ways for general n. We will analyze the rightmost squares of the $3 \times n$ rectangle. Let the rectangle have height 3 and length n.

If the rightmost piece is a 1×3 domino oriented such that the height is 3, then in this case we simply need the number of ways to tile a $3 \times (n-1)$ rectangle. If instead it is placed such that the length is 3, there must also be two more 1×3 dominos of that orientation placed on the right side, reducing the remaining rectangle to $3 \times (n-3)$. So, we see that $a_n = a_{n-1} + a_{n-3}$.

We also have the base cases of $a_1 = 1$, $a_2 = 1$, and $a_3 = 2$. Then we can calculate $\boxed{a_{10} = 28}$ and $\boxed{a_{15} = 189}$.

Solution 8

We will instead count the number of subsets which do not contain any three consecutive numbers using recursion. Let x_n represent this quantity for subsets of $\{1, \dots, n\}$.

We casework on whether n is included or not. If it is not included, then there are simply a_{n-1} ways to finish choosing the subset. If it is included, there are two cases: if $n-1$ is in the subset as well, then $n-2$ cannot be included so then we must simply choose a subset of $\{1, \dots, n-3\}$ under the original constraints. There are a_{n-3} ways. If $n-1$ is not included, then we need to choose a subset of $\{1, \dots, n-2\}$ under the original constraints, so there are a_{n-2} ways to do this. In total,

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}.$$

We also have the base cases of $a_0 = 1$, $a_1 = 2$, and $a_2 = 4$. Then we can calculate up to $a_{10} = 504$ so the number of subsets with three consecutive numbers is $2^{10} - 504 = \boxed{520}$.

Solution 9

First we count the number of ways to satisfy all the constraints except for the one that she must do all three activities at least once. Then we will subtract some cases at the end.

Let $s_L(n)$ be the number of ways to schedule n days with a land sport on the nth day, and let $s_W(n)$ be the number of ways with a water sport on the nth day. Then, we can write the following relations:

$$s_L(n) = s_L(n-1) + s_W(n-1)$$
$$s_W(n) = 2s_L(n-1) + s_W(n-1)$$

The first equation is more straightforward; for the second, if the last sport is a water sport, there are two ways to pick which one it is. In each of those cases, the $(n-1)$th day being a land sport is allowed, so this is the $2s_L(n-1)$ term. In the case that the $(n-1)$th day is a water sport: if the nth day is kayaking, the $(n-1)$th day must be sailing, and if nth day was sailing, then $(n-1)$th day must be kayaking. Thus in total, each of the $s_W(n-1)$ schedules should be counted once in $s_W(n)$.

The base cases are $s_L(1) = 1$ and $s_W(1) = 2$, so then we can find $s_L(8) + s_W(8) = 1393$.

Now we subtract the cases which do not satisfy the last constraint. This means there are either 2 or 1 activities done in total. If there are 2 activities, they must be a land sport and one of the water sports (2 options). So there are $2(2^8)$ ways for this, but this double counts the case in which all 8 days are land sports. So the amount of bad cases is $2(2^8) - 1$, so the answer is

$$1393 - (2(2^8) - 1) = \boxed{882}.$$

Solution 10

We will reduce the number of states the ant can be at after t steps from 5 to 3 by considering the 'distance' of the ant from A. That is, A is at distance 0, B and E are at distance 1, and C and D are at distance 2. Let $p_d(t)$ represent the probability that the ant is at a distance of d after t steps. Note that $p_0(0) = 1$, and $p_1(0) = p_2(0) = 0$.

If the ant is at A, it either stays at A with probability $\frac{1}{3}$ or goes to distance 1 with probability $\frac{2}{3}$. This contributes a term of $\frac{1}{3}p_0(t-1)$ to the expression of $p_0(t)$, and a term of $\frac{2}{3}p_0(t-1)$ to $p_1(t)$. Carrying out similar analysis of the other states, we have:

$$\begin{aligned}
p_0(t) &= \frac{1}{3}p_0(t-1) + \frac{1}{3}p_1(t-1), \\
p_1(t) &= \frac{2}{3}p_0(t-1) + \frac{1}{3}p_1(t-1) + \frac{1}{3}p_2(t-1), \\
p_2(t) &= \frac{1}{3}p_1(t-1) + \frac{2}{3}p_2(t-1).
\end{aligned}$$

Now we simply compute from $t = 0$ to 5 and find that $p_0(5) = \boxed{\frac{53}{243}}$.

Solution 11

Let p_i be the probability of the frog escaping if it is currently on lilypad i. Note that $p_0 = 0$ and $p_5 = 1$. Also, note that by symmetry, $p_1 = 1 - p_4$ and $p_2 = 1 - p_3$. Then, we can also write the following:

$$\begin{aligned}
p_1 &= \frac{1}{5}p_0 + \frac{4}{5}p_2, \\
p_2 &= \frac{2}{5}p_1 + \frac{3}{5}p_3.
\end{aligned}$$

Since $p_0 = 0$, the first equation becomes $p_1 = \frac{4}{5}p_2$. Then plugging this and $p_3 = 1 - p_2$ into the second equation gives

$$p_2 = \frac{2}{5} \cdot \frac{4}{5}p_2 + \frac{3}{5}(1 - p_2) \implies p_2 = \boxed{\frac{15}{32}}.$$

Solution 12

We will prove a more general claim, that for any integer n, there exists an index m such that $n \mid x_m$.

Consider the sequence of pairs of the form (x_i, x_{i+1}) taken modulo n. We claim that this sequence is periodic. Let the sequence be s with $s_i \coloneqq (x_i, x_{i+1}) \pmod{n}$.

First note that s_{i+1} is determined by s_i, since $s_{i+1} = (x_{i+1}, x_i + x_{i+1}) \pmod{n}$. Also, there are only n^2 possible values for an s_i, and since this is an infinite sequence, there must exist some repeated value. Then since s_i fully determines s_{i+1} on its own, one can conclude that the sequence is indeed periodic.

Finally, note that if we extend the sequence to indices before 1, we have $x_0 = 3, x_{-1} = 2, x_{-2} = 1, x_{-3} = 1$, and finally, $x_{-4} = 0$. So, we have $s_{-4} = (0, 1)$. Then, since this pair including a 0 must appear again later in the sequence, we can in fact find infinitely many m such that $x_m \equiv 0 \pmod{n}$, as desired. $\square$