ORMC Olympiad Group
Week 1 Solutions

Sumith Nalabolu

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Note: In some solutions, I use summation notation. That is, I will use something like the following:

$$\sum_{i=1}^{n} a_i$$

which represents the sum $a_1 + a_2 + \ldots + a_n$.

Solutions

Solution 1

Following the strategy covered in class, we know the formula for S_n^4 will be a polynomial p of degree 5 such that $p(n) - p(n-1) = n^4$. Let $p(n) = an^5 + bn^4 + cn^3 + dn^2 + en$ (we can set the constant term to be 0). Then, we have

$$\begin{aligned}
&n^4 = p(n) - p(n-1) \\
\implies &n^4 = (an^5 + bn^4 + cn^3 + dn^2 + en) - (a(n-1)^5 + b(n-1)^4 + c(n-1)^3 + d(n-1)^2 + e(n-1)) \\
\implies &n^4 = n^4(5a) + n^3(4b - 10a) + n^2(3c - 6b + 10a) + n(2d - 3c + 4b - 5a) + (e - d + c - b + a).
\end{aligned}$$

So, we compare coefficients in order and find that

$$a = \frac{1}{5}, b = \frac{1}{2}, c = \frac{1}{3}, d = 0, e = -\frac{1}{30}.$$

So the formula is $S_n^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$.

Solution 2

Let a_n be the amount of squares in figure n. Notice that to get from figure n to $n+1$, we add $4n$ squares around the perimeter. Thus,

$$a_n = 1 + 4 \cdot 1 + 4 \cdot 2 + \ldots + 4 \cdot n = 1 + 4\left(\frac{n(n+1)}{2}\right) = 1 + 2n(n+1).$$

So the amount of squares in figure 100 is $\boxed{20201}$.

Solution 3

(a) The pattern is that we add 6, then 12, then 18, then 24, so the next must be adding 30. So the next term is $61 + 30 = \boxed{91}$.

(b) As described above, $a_n = a_{n-1} + 6(n-1)$, if we take $a_1 = 1$ as the first term.

(c) Note that

$$a_n = a_1 + 6 \cdot 1 + 6 \cdot 2 + \ldots + 6 \cdot (n-1).$$

We can then write

$$\begin{aligned}
a_n &= 1 + 6(1 + 2 + \ldots + (n-1)) \\
&= 1 + 6 \cdot \frac{(n-1)(n)}{2} \\
&= \boxed{3n^2 - 3n + 1}.
\end{aligned}$$

(d) Note that $a_n = 3n^2 - 3n + 1 = n^3 - (n-1)^3$, so

$$\sum_{i=1}^{100} a_n = (1^3 - 0^3) + (2^3 - 1^3) + \ldots + (100^3 - 99^3) = 100^3 - 0^3 = 100^3.$$

This factors to $2^6 \cdot 5^6$ and thus has $\boxed{49}$ factors.

Solution 4

We can write each term in the general form of $i(2i-1)$. So, we want to compute $\sum_{i=3}^{13} i(2i-1)$. Now we expand the term and compute the sum by splitting it up:

$$\begin{aligned}
\sum_{i=3}^{13} i(2i-1) &= \sum_{i=3}^{13}(2i^2-i) \\
&= 2\sum_{i=3}^{13} i^2 - \sum_{i=3}^{13} i \\
&= 2\left(\sum_{i=1}^{13} i^2\right) - 2(1^2+2^2) - \left(\sum_{i=1}^{13} i\right) + (1+2) \\
&= 2\cdot\frac{13(14)(27)}{6} - 10 - \frac{13(14)}{2} + 3 \\
&= 13(7)\left(\frac{4(27)}{6} - 1\right) - 10 + 3 \\
&= \boxed{1540}.
\end{aligned}$$

Solution 5

We can rewrite the desired sum as follows by considering the missing terms (the squares of even numbers):

$$\begin{aligned}
1^2+3^2+\ldots+(2n-1)^2 &= (1^2+2^2+3^2+\ldots+(2n-1)^2) - (2^2+4^2+\ldots+(2n-2)^2) \\
&= (1^2+2^2+\ldots+(2n-1)^2) - 4(1^2+2^2+\ldots+(n-1)^2).
\end{aligned}$$

Then we can use the summation formulas to simplify this expression to:

$$\begin{aligned}
\frac{(2n-1)(2n)(4n-1)}{6} - 4\cdot\frac{(n-1)(n)(2n-1)}{6} &= \frac{(2n-1)(2n)(4n-1-2(n-1))}{6} \\
&= \frac{2n(2n-1)(2n+1)}{6} \\
&= \frac{n(4n^2-1)}{3}.
\end{aligned}$$

Then the answer to part (a) can be found by plugging in $n = 25$:

$$\frac{25(4\cdot 625-1)}{3} = 25(833) = 25(208\cdot 4)+1 = \boxed{20825}.$$

Solution 6

First sum the terms; we can pair the terms as follows and use difference of squares:

$$1^2 + (3^2-2^2) + (5^2-4^2) + \ldots + (101^2-100^2) = 1 + (3+2)(1) + \ldots + (101+100)(1).$$

Thus the sum is just equal to $1+2+\ldots+101$, and the answer is then this divided by 101; so, the answer is

$$\frac{1}{101}\cdot\frac{101(102)}{2} = \boxed{51}.$$

Solution 7

Note that these fractions are of the form $k + \frac{j}{5}$ for integer k and $1 \le j \le 4$. For example, $\frac{61}{5}, \frac{62}{5}, \frac{63}{5}, \frac{64}{5}$ are the ones between 12 and 13.

If we first sum these 4 fractions with a certain integer part k, we get $4k + \frac{1+2+3+4}{5} = 4k + 2$. Thus, the answer is just

$$\sum_{k=12}^{20} 4k + 2.$$

The +2 term evaluates to 18 in this sum; for the other term, note that

$$(12 + \ldots + 20) = (1 + \ldots + 20) - (1 + \ldots + 11) = \frac{20(21) - 11(12)}{2} = 144.$$

So the answer is $4(144) + 18 = \boxed{594}$.

Solution 8

We follow the same procedure as problem 1. The expression must be a polynomial $p(n)$ of degree 6 satisfying $p(n) - p(n-1) = n^5$. So, let $p(n) = an^6 + bn^5 + cn^4 + dn^3 + en^2 + fn$, and then we have:

$$\begin{aligned}
& n^5 = p(n) - p(n-1) \\
\implies & n^5 = (an^6 + bn^5 + cn^4 + dn^3 + en^2 + fn) - (a(n-1)^6 + b(n-1)^5 + c(n-1)^4 + d(n-1)^3 + e(n-1)^2 + f(n-1)) \\
\implies & n^5 = n^5(6a) + n^4(5b - 15a) + n^3(4c - 10b + 20a) + n^2(3d - 6c + 10b - 15a) + n(2e - 3d + 4c - 5b + 6a) \\
& \qquad + (f - e + d - c + b - a).
\end{aligned}$$

From comparing each coefficient from n^5 to the constant coefficient, we can compute

$$a = \frac{1}{6}, b = \frac{1}{2}, c = \frac{5}{12}, d = 0, e = \frac{1}{12}, f = 0.$$

So, the answer is $S_n^5 = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 + \frac{1}{12}n^2$.

Solution 9

Define this sum as S. Also define $T = 2^2 + 5^2 + \ldots + 101^2$. Then note that

$$\begin{aligned}
S + T &= 1^2 + 2^2 + 4^2 + 5^2 + \ldots + 100^2 + 101^2 \\
&= (1^2 + 2^2 + \ldots + 102^2) - (3^2 + 6^2 + \ldots + 102^2) \\
&= (1^2 + 2^2 + \ldots + 102^2) - 9(1^2 + 2^2 + \ldots 34^2)
\end{aligned}$$

and

$$\begin{aligned}
T - S &= (2^2 - 1^2) + (5^2 - 4^2) + \ldots + (101^2 - 100^2) \\
&= 1 + 2 + 4 + 5 + \ldots + 100 + 101 \\
&= (1 + 2 + \ldots + 102) - (3 + 6 + \ldots + 102) \\
&= (1 + 2 + \ldots + 102) - 3(1 + 2 + \ldots + 34).
\end{aligned}$$

Then note that $S = ((S+T) - (T-S))/2$, and we can compute both $S+T$ and $T-S$ using our summation formulas:

$$S + T = \frac{102(103)(205) - 9(34)(35)(69)}{6},$$

$$T - S = \frac{102(103) - 3(34)(35)}{2}.$$

Then we can compute $((S+T) - (T-S))/2 = 116161 \implies \boxed{161}$.

Note on computation: the expression is 1/2 times the following:

$$\frac{102(103)(205) - 9(34)(35)(69)}{6} - \frac{102(103) - 3(34)(35)}{2},$$

but this can be simplified as follows by finding common factors in terms:

$$102(103)\left(\frac{205}{6} - \frac{1}{2}\right) + 34(35)\left(-\frac{9(69)}{6} + \frac{3}{2}\right)$$

$$\implies 102(103)\frac{101}{3} + 34(35)(-102)$$

$$\implies 34\left((103)(101) - 34(35)(3)\right)$$

which can be computed more easily: we wanted half of the above expression, which is $17((103)(101) - 34(35)(3))$. Then since we are taking $\pmod{1000}$, it is

$$17(10403 - 3570) \equiv 17(-167) \equiv -2839 \equiv \boxed{161} \pmod{1000}.$$

Solution 10

(a) $\boxed{248}$.

(b) Define $a_1 = 1$; then the pattern is $a_n = 2a_{n-1} + (n-2)$.

(c) Define another sequence $b_n = a_n + n$. Then from plugging in the recurrence for a_n, we have

$$b_n = a_n + n \implies b_n = (2a_{n-1} + (n-2)) + n = 2a_{n-1} + 2n - 2.$$

But also, we have $b_{n-1} = a_{n-1} + (n-1)$. So we can substitute $a_n = b_{n-1} - (n-1)$:

$$b_n = 2(b_{n-1} - (n-1)) + 2n - 2 \implies b_n = 2b_{n-1}$$

after simplification. Then since $b_1 = 2$, $b_n = 2^n$. So $\boxed{a_n = 2^n - n}$.

(d) Note that $2^0 + 2^1 + \ldots 2^{n-1} = 2^n - 1$, by sum of geometric series formula. So, the desired sum is

$$\begin{aligned}(2^1 - 1) + (2^2 - 2) + \ldots (2^{99} - 99) &= (2^1 + \ldots + 2^{99}) - (1 + \ldots + 99) \\ &= (2^{100} - 2) - \frac{99(100)}{2} \\ &= 2^{100} - 4952.\end{aligned}$$

Notice the term is of the form $2^n - 2$ because the sum of powers of 2 does not include the 2^0 term.

(e) Following the same strategy as above, the answer is $2^{n+1} - 2 - \dfrac{n(n+1)}{2}$.