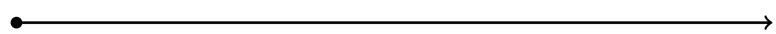
Oleg Gleizer LAMC

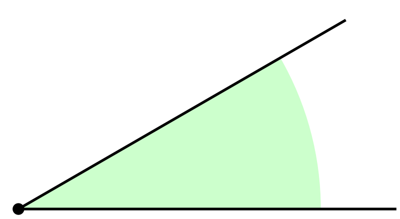
Introduction to Geometry Lesson 2

Angles

A *ray* is a half of a straight line, finite in one direction, but infinite in the other. A ray contains its boundary point.



A (plane) *angle* is a plane figure formed by two rays with a common vertex and by all the points in between.



Looking at the above picture, you have to imagine that each of the two rays goes to infinity and so does the green coloring. This way, the rays split the plane into two parts, the green and white. The green-colored part, together with the boundary rays, is an angle. The white-colored region, bounded by the same rays, is also an angle.

If the rays forming an angle coincide, the smaller one of the two is called a *zero angle*. The angle complementing it to a full plane is called a *full angle*.

If the rays forming an angle lie on a straight line, the angle is called a *straight angle*.

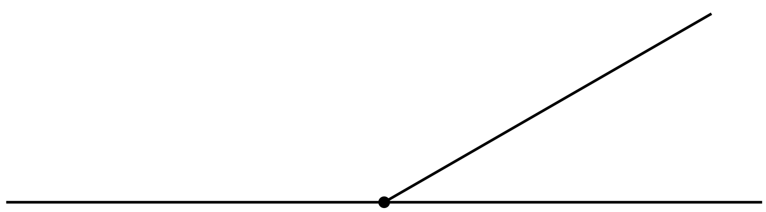
You have to imagine that the entire upper half-plane on the above picture is colored green.

The word *congruent* means *coincide when superimposed*. If two geometric figures are congruent, then all of their features, lengths of the corresponding sides, sizes of the corresponding angles, etc. are exactly the same.

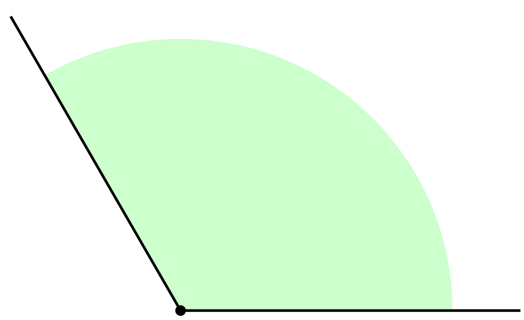
Problem 1 *Define a straight angle in a way different from the above. Hint: consider the angle complementing it to a full angle.*

Problem 2 *Draw a straight line in the space below. How can you make it into a straight angle?*

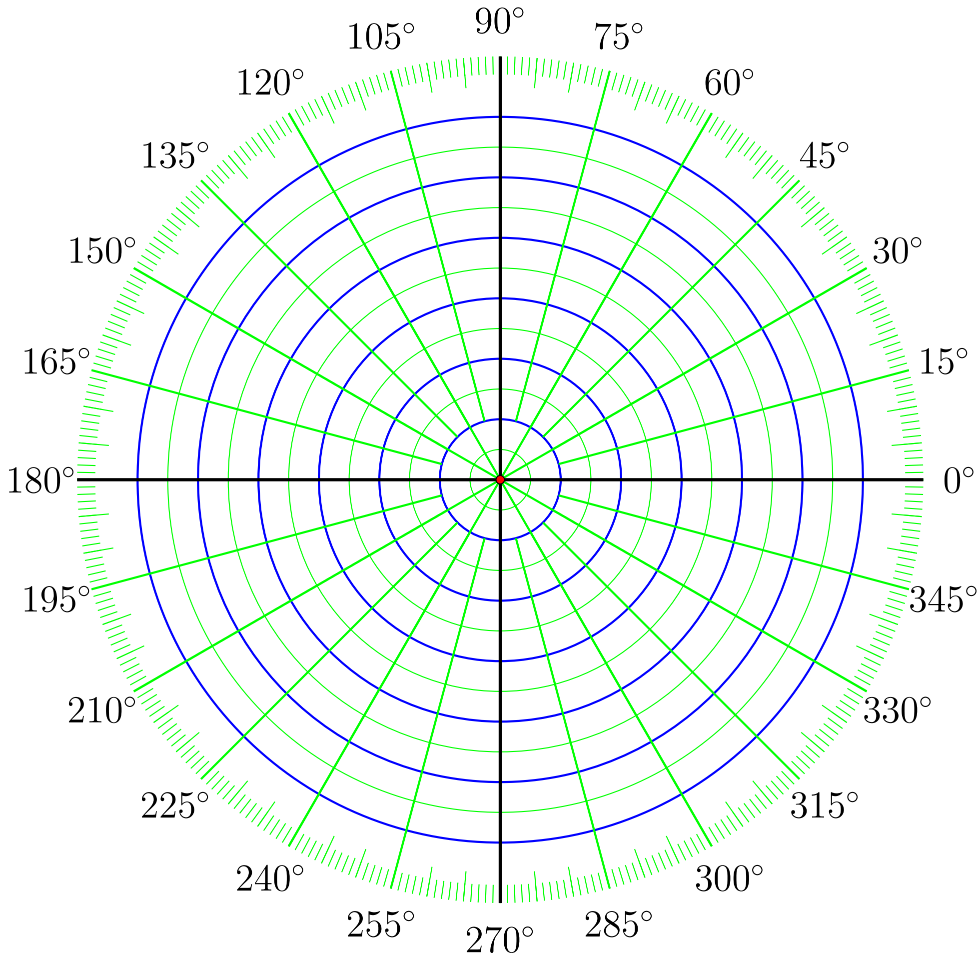
Two angles are called *supplementary*, if they add up to a straight angle.



Problem 3 *Using a ruler, draw an angle supplementary to the green-colored angle below. How many ways are there to solve the problem?*



A standard way to measure an angle is to divide a full angle into 360 equal parts. One part is called one (angular) *degree* and is denoted as 1°. A circumference bearing the 1° marks, or a part of it, is called a *protractor*.

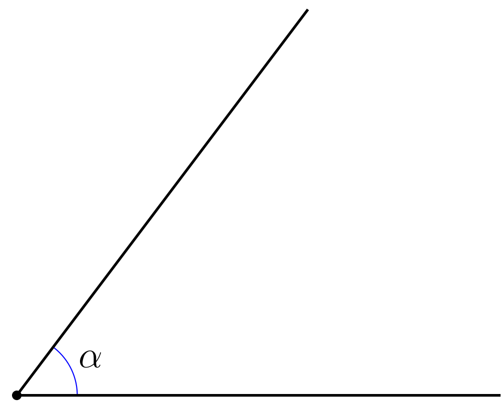


Question 1 *What is the size, in degrees, of a straight angle?*

It is customary to use lower-case Greek letters to denote angles and their sizes. You will find the table with letters of the Greek alphabet on page 25 of this handout.

Problem 4 *Use degrees to write an algebraic statement showing that angles α and β are supplementary.*

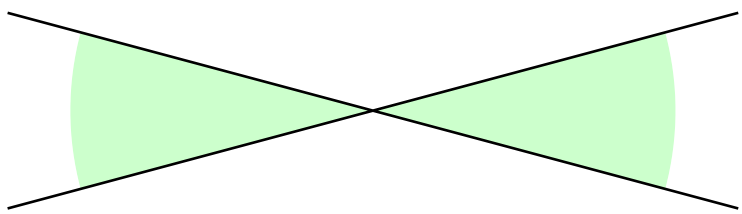
Problem 5 *Use a protractor to measure the following angle. Round to the nearest degree.*



$\alpha =$

Problem 6 *Use a ruler and a protractor to draw a 75° angle in the space below.*

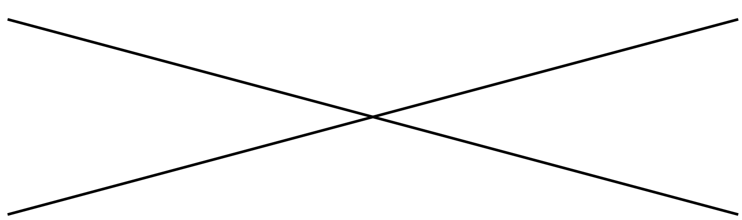
Two angles are called *vertical* or *opposite*, if they have a common vertex and their four side rays form two straight lines as on the picture below.



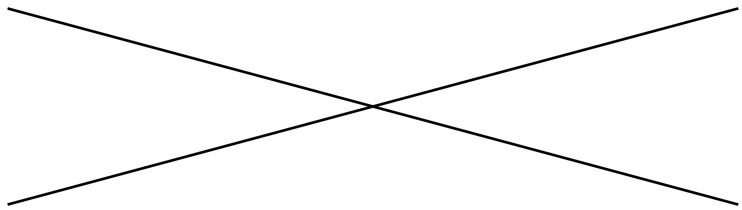
The green-colored angles at the above picture are vertical. The white-colored angles, supplementary to the green-colored angles, are vertical, too.

Proposition 1 *Vertical angles are congruent.*

Problem 7 *Prove Proposition 1 by consider a rotation of one of the angles around its vertex by a straight angle.*

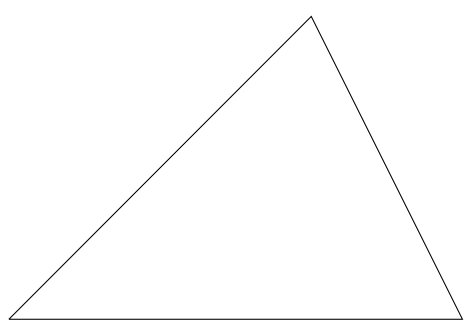


Problem 8 *Give an algebraic proof to Proposition 1.*



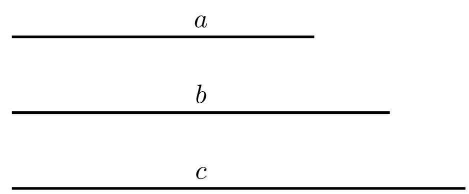
Triangles

The word *polygon* means *multi-angled* in the language of Euclid, the ancient Greek. The simplest possible polygon in the Euclidean plane is a triangle, a polygon with three vertices, sides, and angles.



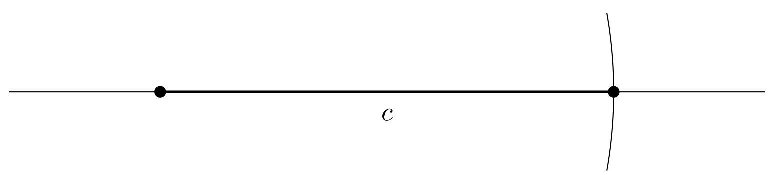
Let us learn various ways to construct a triangle using a compass and a ruler.

Example 1 (SSS) *Using a compass and a ruler, construct a triangle having the given sides,* a*,* b*, and* c*.*



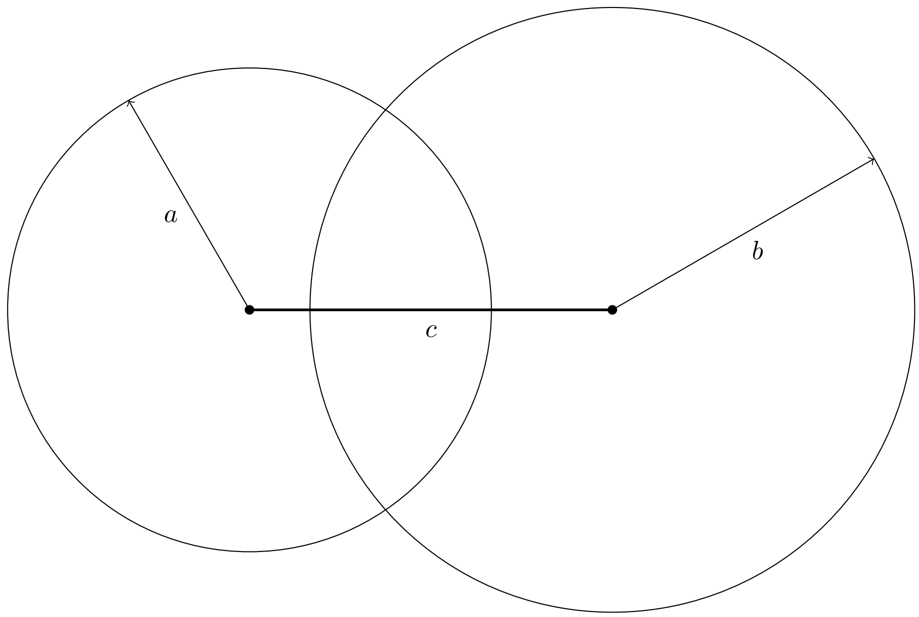
Step 1. Draw a straight line and mark a point on it.

Step 2. Measure side c *with a compass, place the needle at the marked point on the line, and mark side* c *on the line.*

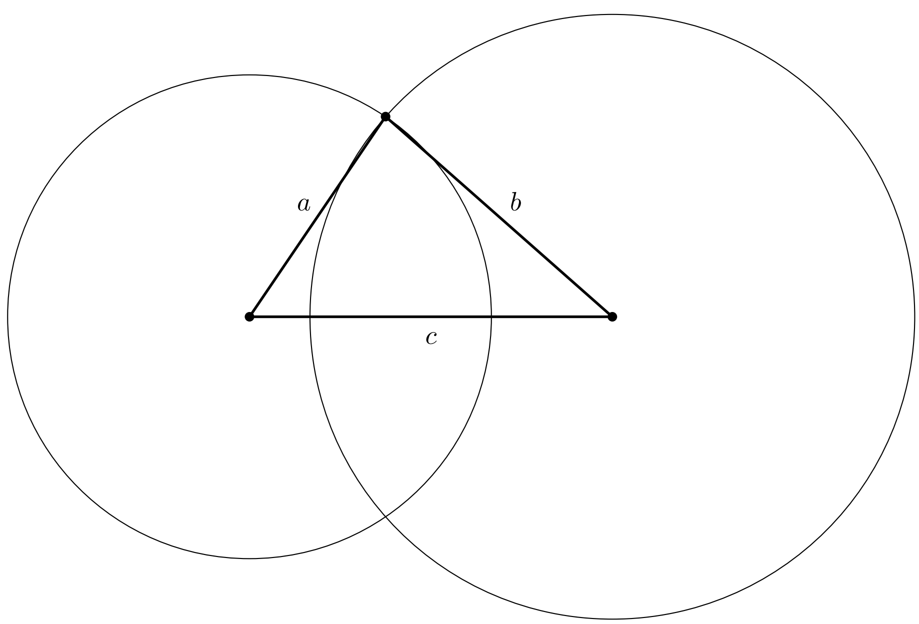


Allowing a bit of ambiguity, we will use the same notation for a side and its length.

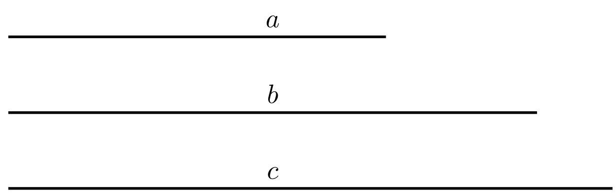
Step 3. Now is the time to recall the definition of a circumference. The points of the plane having the distance a *from the left endpoint of the side* c *form a circumference of radius* a *centered at the point. Similarly, the points having the distance* b *from the right endpoint of* c *belong to the circumference of radius* b *centered at the latter point.*



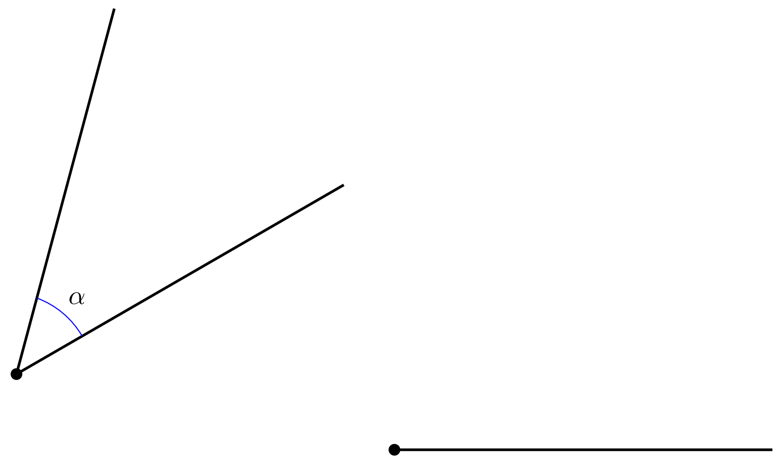
Step 4. The circumferences intersect at two points. Each of them has the distance a from the left endpoint of the side c and the distance b from the right endpoint. We can pick either one as the third vertex of the triangle.



Problem 9 *Use a compass and a ruler to construct a triangle having the following sides in the space below.*



Example 2 *Using a compass and a ruler, construct the given angle α having the given ray as its side.*

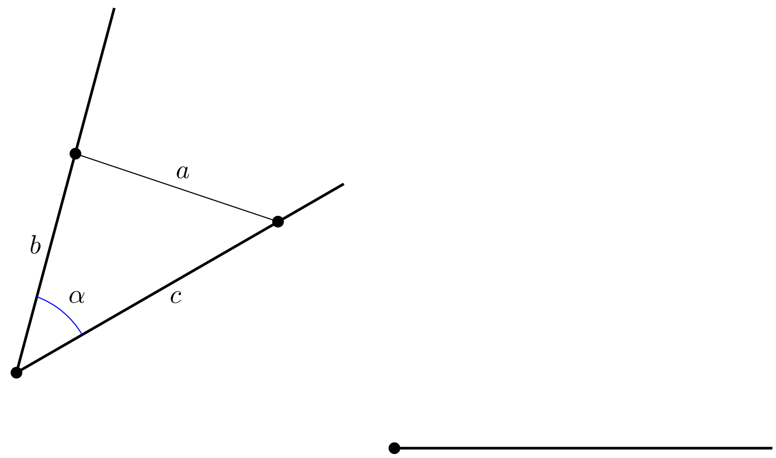


Below, you will see two different solutions to this problem, each having its own merit.

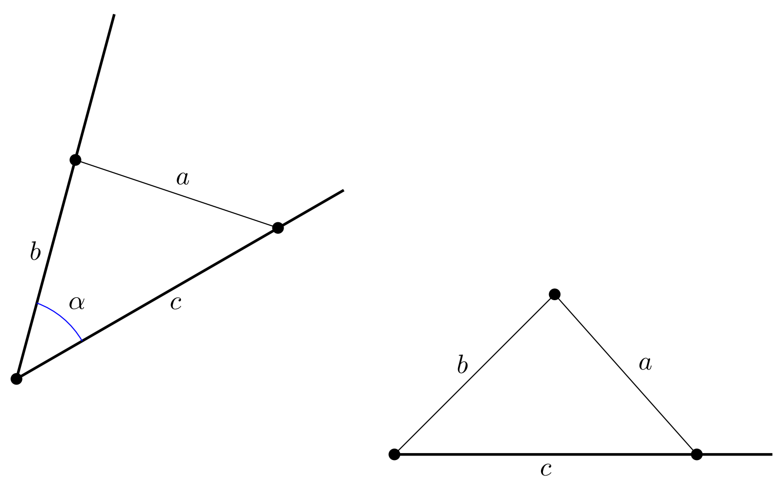
Solution 1. *The following approach is widely used in Mathematics. We need to solve a problem, but we don't know how. Instead, let us solve a problem we know how to solve. In this particular case, we already know how to construct a triangle with given sides. Instead of solving the original problem, let's solve this one!*

The word "auxiliary" means "providing supplementary or additional help and support" as in an "auxiliary nanny", a nanny occasionally employed in addition to the main one. Like many more words used in science, this one originates from Latin, the language of the ancient Rome. Its progenitor, the Latin word "auxilium" means "help".

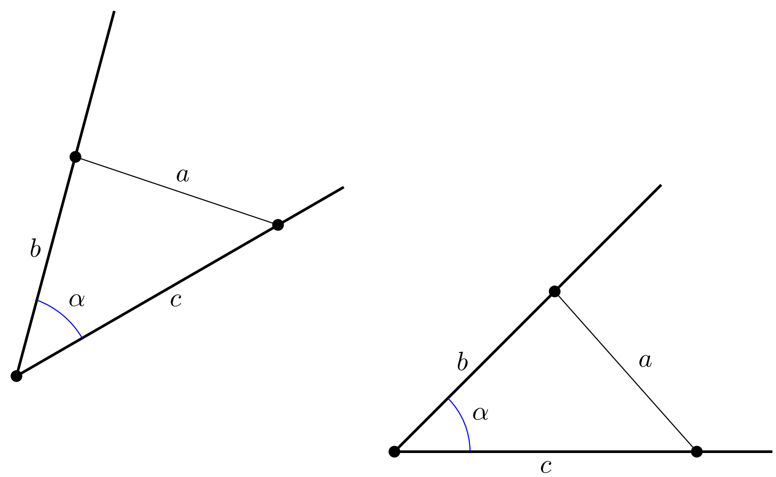
Step 1. Let us draw an auxiliary triangle having α as its angle. Traditionally, the side opposite to α is called a.



Step 2. Using the method of Example 1, let us construct the triangle having the sides a, b, and c so that the side c belongs to the given ray and its left endpoint coincides with the ray's vertex.



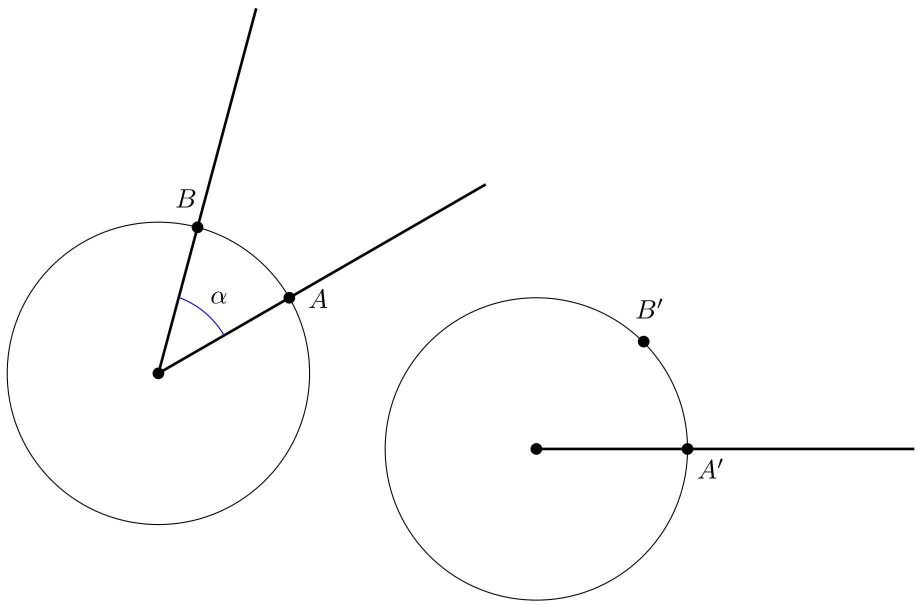
Since the two constructed triangles are congruent, the angles α opposite to the sides a must be congruent, too.



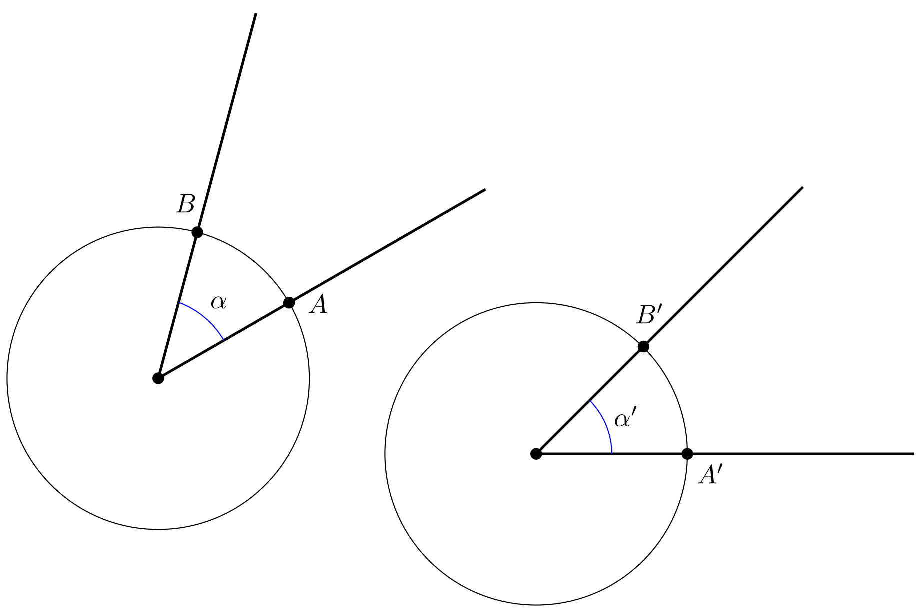
Solution 2. *Let us draw a circumference centered at the vertex of the original angle. The sides of the angle mark two points, A and B, on the line.*

The notation $\overline{AB}$ means "a straight line segment with the endpoints A and B". The notation AB means the length of the segment.

Let us draw another circumference with the same radius centered at the vertex of the given ray. Let us call A' the point where the circumference intersects the ray. Let us measure the distance between the points A and B with a compass. Let us further stick the compass's needle at A' and mark the point B' lying on the second circumference such that $AB = A'B'$.



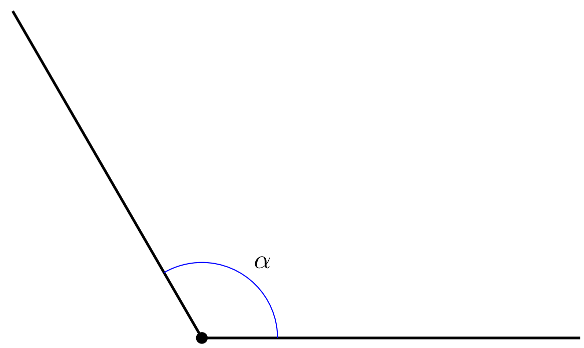
The last thing to do is to draw the ray originating at the center of the second circumference and passing through B'.

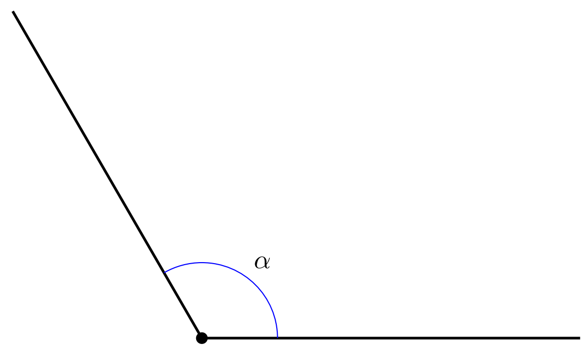


The sign $\cong$ means "congruent".

Problem 10 *Prove that* $\alpha \cong \alpha'$.

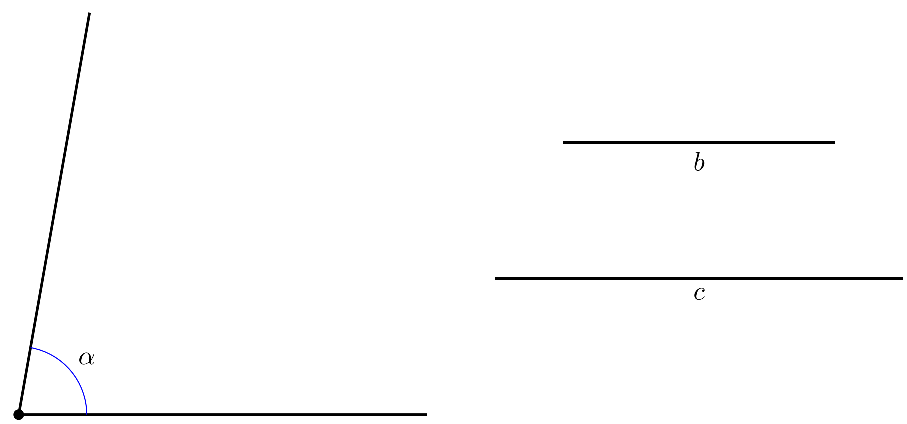
Problem 11 *Use a compass and a ruler to construct an angle congruent to the angle* α *below in two different ways. Use an auxiliary triangle on this page and a circumference on the next one.*





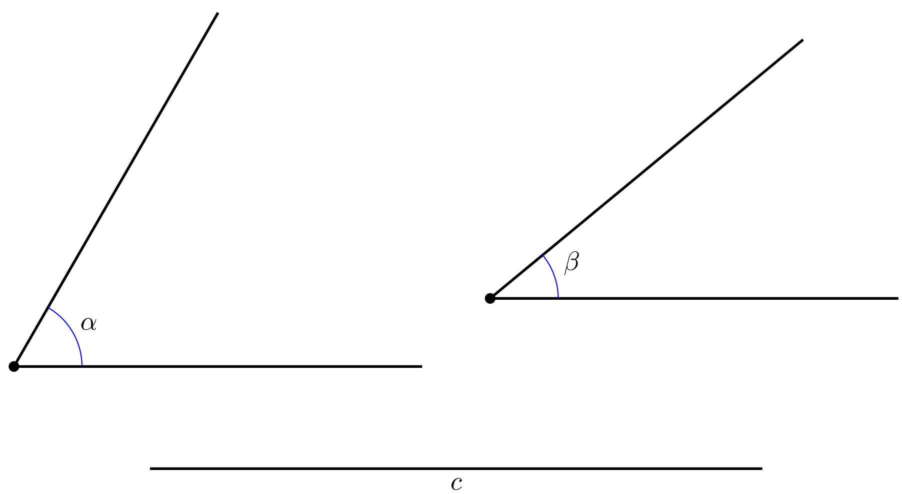
The word “adjacent” means “lying near to” as in “adjacent rooms” or “the houses adjacent to the park”. It was inherited from Latin without a change in spelling.

Problem 12 (SAS) *In the space at the bottom of the page, construct a triangle with the angle α and with the adjacent sides b and c given below. (In this case, the word "adjacent" means that the vertex of α is an endpoint of the sides b and c).*



Hint: begin with drawing the angle. If you choose the auxiliary triangle method for constructing the latter, you can use not some arbitrary sides b and c, but the given ones right away!

Problem 13 (ASA) *Construct a triangle with the side c and adjacent angles α and β given below. (In this case, the word "adjacent" means that the endpoints of c are the vertices of α and β.)*



Note that by carrying out the above constructions, we just have (nearly) proven the following very important theorem.

Theorem 1 *Two triangles in the Euclidean plane are congruent if either of the following holds.*

- **SSS** *(Example 1) The sides are pairwise congruent (or the side lengths are pairwise equal).*

$$a \cong a',\ b \cong b',\ c \cong c'$$

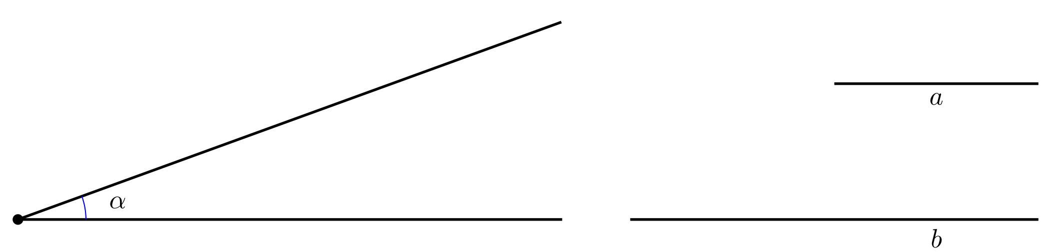
- **SAS** *(Problem 12) The triangles have congruent angles and the sides adjacent to the angles are pairwise congruent.*

$$b \cong b',\ \alpha \cong \alpha',\ c \cong c'$$

- **ASA** *(Problem 13) The triangles have congruent sides , and the adjacent angles are pairwise congruent.*

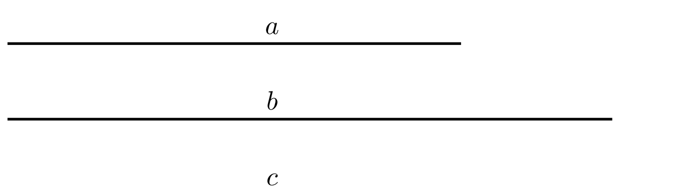
$$\alpha \cong \alpha',\ c \cong c', \beta = \beta'$$

Problem 14 **(SSA)** *At the top of the next page, construct a triangle with the angle α given below as well as with the side b adjacent to α and with the side a opposite to the angle. How many non-congruent solutions do you get?*

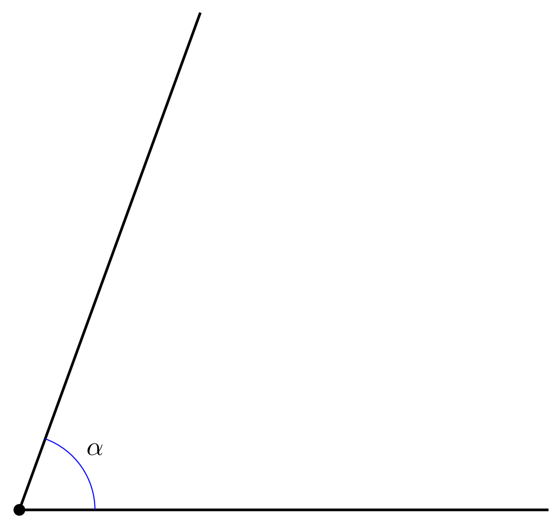


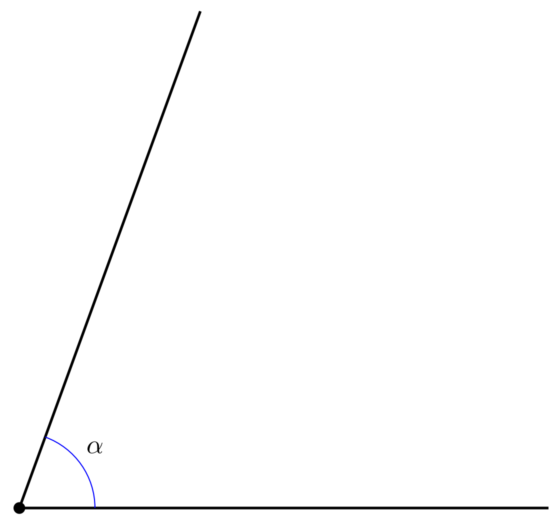
Note 1 *Unlike SSS, SAS, and ASA, SSA can produce non-congruent triangles!*

Homework Problem 1 *Use a compass and a ruler to construct a triangle having the following sides.*



Homework Problem 2 *Use a compass and a ruler to construct an angle congruent to the angle α below in two different ways. Use an auxiliary triangle on this page and an auxiliary circumferences on the next one.*

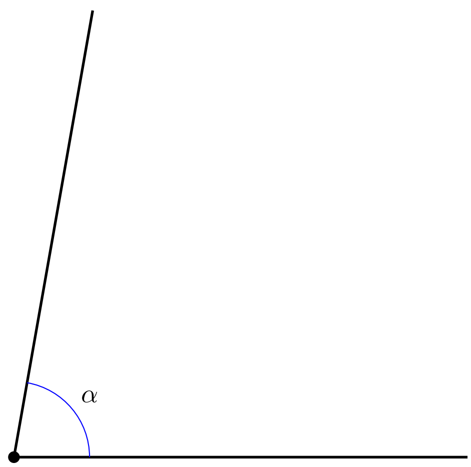


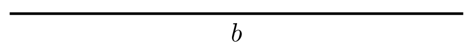


Which method do you like more? Why?

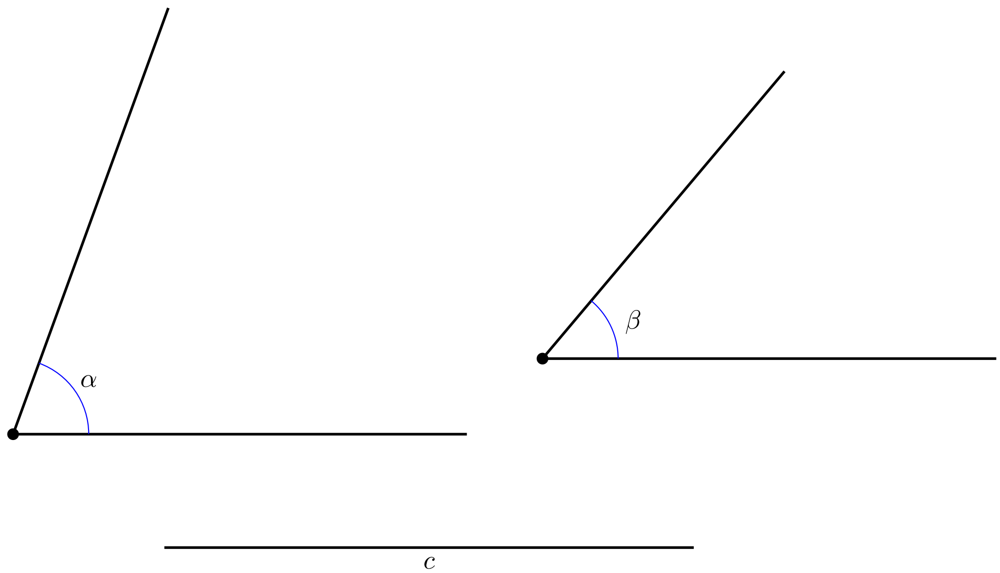
A triangle is called *isosceles*, if it has two sides of equal length.

Homework Problem 3 *In the space below, construct an isosceles triangle with the angle α and with the sides adjacent to it congruent to the segment b.*





Homework Problem 4 *Construct a triangle with the side* c *and with adjacent angles* α *and* β *given below.*



A triangle is called *equilateral* if all of its sides have equal length.

Homework Problem 5 *In the space below, construct an equilateral triangle with* $2''$ *sides.*

Greek alphabet

You will find the Greek alphabet in the table below.

Letter	$A\ \alpha$	$B\ \beta$	$\Gamma\ \gamma$	$\Delta\ \delta$	$E\ \epsilon$	$Z\ \zeta$	$H\ \eta$	$\Theta\ \theta$
Name	alpha	beta	gamma	delta	epsilon	zeta	eta	theta
Letter	$I\ \iota$	$K\ \kappa$	$\Lambda\ \lambda$	$M\ \mu$	$N\ \nu$	$\Xi\ \xi$	$O\ o$	$\Pi\ \pi$
Name	iota	kappa	lambda	mu	nu	xi	omicron	pi
Letter	$P\ \rho$	$\Sigma\ \sigma$	$T\ \tau$	$Y\ \upsilon$	$\Phi\ \phi$	$X\ \chi$	$\Psi\ \psi$	$\Omega\ \omega$
Name	rho	sigma	tau	upsilon	phi	chi	psi	omega

Question 2 *What did the word "alphabet" originate from?*

Question 3 *What is the meaning of the expression "from alpha to omega"?*

Self-test questions

- What is a ray?
- What is an angle?
- What is a straight angle?
- What is an angle complementary to the given one?
- What is an angle supplementary to the given one?
- What geometric figures do we call congruent?
- What is $1°$?
- How many degrees are there in a full angle? In a straight angle?
- What angles are called vertical?
- Are vertical angles congruent? Why or why not?
- What is the meaning of the word *polygon*?
- Can congruent triangles have sides of different length?
- How can one construct a triangle with given sides using a compass and straightedge as tools?
- How can one construct an angle congruent to the given one

using a compass and straightedge as tools?

- What is the meaning of the word *adjacent*?

- How can one construct a triangle with a compass and straightedge, given its angle and two adjacent sides?

- How can one construct a triangle with a compass and straightedge, given its two angles and the side adjacent to both of them?

- Formulate the **SSS, SAS, ASA** theorem.

- Does the **SSA** construction always produce a triangle congruent to the given one? Why or why not?