ORMC Olympiad Group
Week 2

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Recursion

Lecture Notes

The idea behind (combinatorial) recursion is to represent the $f(n)$ in terms of previous terms $f(n-1), f(n-2), \ldots, f(1)$. Most of the time, $f(n)$ represents the number of solutions for a particular problem. The tricky part is to create the recursive relation from the combinatorial problem. For these kind of problems, the first step always has to be correctly generalizing the problem.

Example 1. How many ways we can cover 1×10 rectangle completely by 1×1 or 1×2 dominoes?

Solution: We start this problem by **generalizing** as follows: **Define s_n as the number of ways we can cover $1 \times n$ rectangle completely by 1×1 or 1×2 dominoes.** Then this problem is actually asking for s_{10}.

Example 2. **(AMC12 2009B)** How many sequences of 0s and 1s of length 19 are there that begin with a 0, end with a 0, contain no two consecutive 0s, and contain no three consecutive 1s?

(A) 55 **(B)** 60 **(C)** 65 **(D)** 70 **(E)** 75

Example 3. **(AMC12 2007A P25)** Call a set of integers spacy if it contains no more than one out of any three consecutive integers. How many subsets

of $\{1, 2, 3, \ldots, 12\}$, including the empty set, are spacy?
HINT: Define a_n as the number of spacy subsets of $\{1, 2, 3, \ldots, n\}$. Then this question asks a_{12}. You need to derive a recursive relation on a_n.

We can also use recursion in probability problems.

Example 4. Let $ABCD$ be a regular tetrahedron. A rabbit starts on the vertex A, and at each step, she chooses one of the three other vertices uniformly randomly and jumps to that vertex. The probability that the rabbit comes back to vertex A in 6 steps can be represented as a fraction $\frac{m}{n}$ where m, n are relatively prime positive integers. Find m.

Problems

1. (a) How many ways we can cover 1×10 rectangle completely using 1×1, 1×2 and 1×3 dominoes?

   (b) How many ways we can cover 1×10 rectangle completely using 1×1 and 1×3 dominoes?

2. Find the number of ways that we can cover $2 - by - 10$ rectangle with 1×2 and 2×2 dominos.

3. Let ABC be an equilateral triangle. An ant is staying at vertex A in the beginning. At every step, the ant walks to one of the other two vertices with equal probability. For example, in the first step, the ant can go B with 0.5 probability, and can go C with with 0.5 probability. What is the probability that the ant will be at vertex A after exactly 6 steps?

4. Let ABC be an equilateral triangle. An ant is staying at vertex A in the beginning. At every step, the ant walks to one of the other two vertices. Find the number of different paths so that in 8 steps the ant comes back to vertex A.

5. A frog tries to go upstairs using stairway. He can jump 2 or 3 steps at once. If the stairway consist of 19 steps, in how many ways the frog can jump upstairs?

6. **(AIME 2001I)** A mail carrier delivers mail to the nineteen houses on the east side of Elm Street. The carrier notices that no two adjacent houses ever get mail on the same day, but that there are never more than two houses in a row that get no mail on the same day. How many different patterns of mail delivery are possible?

7. (a) Find the number of ways we can cover 3×10 rectangle by ten 1×3 dominos.

   (b) Find the number of ways we can cover 3×15 rectangle by fifteen 1×3 dominos.

8. How many subsets of $\{1, 2, 3, \ldots, 10\}$ contains three consecutive numbers?

9. **(From MathStackExchange)** Jo is going on an 8-day activity holiday. Each day she can choose one of the water sports: kayaking or sailing, or land-based sports. She never does different water sports on consecutive days. She also wants to try all three options on at least one day of her holiday. How many different schedules are possible?

10. Let $ABCDE$ be a pentagon. An ant is staying at vertex A in the beginning. At every step, the ant has three choices. It can stay where it is with probability $\frac{1}{3}$, or it can walk to one of the two neighbor vertices with probability $\frac{1}{3}$. Note that we call two vertices neighbor if they are connected through an edge. For example, in the first step, the ant can stay at A with $\frac{1}{3}$ probability, or it can go to B with $\frac{1}{3}$ probability, or it can go to E with with $\frac{1}{3}$ probability. What is the probability that the ant will be at vertex A after exactly 5 steps?

11. **(AMC10-2014B P25-modified)** In a small pond there are 6 lily pads in a row labeled 0 through 5. A frog is sitting on pad 2. When the frog is on pad N, $0 < N < 5$, it will jump to pad $N-1$ with probability $\frac{N}{5}$ and to pad $N+1$ with probability $\frac{5-N}{5}$. Each jump is independent of the previous jumps. If the frog reaches pad 0 it will be eaten by a patiently waiting snake. If the frog reaches pad 5 it will exit the pond, never to return. What is the probability that the frog will escape without being eaten by the snake?

12. Define the sequence $x_1 = 5, x_2 = 8$ and $x_n = x_{n-1} + x_{n-2}$ for $n \geq 3$. Prove that there is an index m such that $2021^{1919} | x_m$