LAMC Junior Circle January 29, 2012

Olga Radko and Oleg Gleizer

A digit and a number. Place-value numerals. Properties of numbers.

The numeral system we use has ten digits,

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9,$$

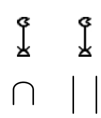
and infinitely many numbers. The numbers are made of digits somewhat like the words of the English language are made of letters. However, there is a big difference. A letter has one and the same meaning no matter what its location in the word is. It may be pronounced in different ways, but still an *a* means an *a*, whether located at the beginning of the word, as in *apple*, or in the middle, as in *mural*. A digit can have a different value depending on its position in a number. For example, 7 means seven in a single digit number, but it means seven million in the number 987,654,321. The numeral systems having this property are called place-value. Ours has ten digits, so its full name is a *place-value decimal system*.

Problem 1 *What is the value of 5 in the number 7,541?*

Place-value numerals are a great invention of the human kind that makes computations simpler to perform. To appreciate this simplicity, let us take a look at the numeral system used in the ancient Egypt.

1	10	100	1,000	10,000	100,000	1,000,000 or many
|	∩				 or 	

For example, they would have written 2012 as

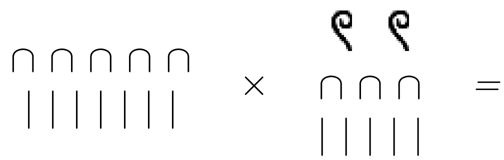


Problem 2 *Compute without switching to the decimal system.*

𓆼𓆼𓎆𓏺𓏺 − 𓍢𓍢𓍢𓍢𓍢𓍢𓍢𓍢𓎆𓎆𓎆𓎆𓎆𓎆𓏺𓏺𓏺𓏺𓏺

Not yet convinced that a place-value system is a great invention? Then please use Egyptian multiplication to multiply the following two numbers. Thanks to the authors having some mercy on little children, you are allowed to use modern numerals to perform Egyptian multiplication. Once finished, please rewrite the answer using Egyptian numerals. Then check your answer using long multiplication (if you are already familiar with the latter).

Problem 3



There are two pillars the convenience of place-value numeral systems is based on. To take a look at the first, let us rewrite the number 3,641 as

$$3,641 = 3 \times 1,000 + 6 \times 100 + 4 \times 10 + 1 \times 1$$

Using the power notations, we can further rewrite this as follows.

$$3,641 = 3 \times 10^3 + 6 \times 10^2 + 4 \times 10^1 + 1 \times 10^0$$

As you can see, we implicitly write numbers as sums of powers of 10. The digits 0,1,2,3,4,5,6,7,8,9 serve as coefficients of the decomposition.

Problem 4 *Express the following numbers as explicit sums of powers of 10.*

$71 =$

$987 =$

$20,000 =$

$1,023,008 =$

The second pillar is the fact that multiplying by ten shifts a number one position forward, like $23 \times 10 = 230$. Since $23 = 2 \times 10^1 + 3 \times 10^0$, we have $23 \times 10 = (2 \times 10^1 + 3 \times 10^0) \times 10 = 2 \times 10^2 + 3 \times 10^1 = 200 + 30 = 230$. This property is actually responsible for our numeral system to be place-value!

Problem 5 *How many positions does a number shift to the left when we multiply it by 1,000? By 100,000? Give an example for each of the above.*

Let us recall *commutativity* of addition, an important property of numbers, whether written in a place-value system or any other.

Problem 6 *1. What does it mean that addition is commutative?*

2. Do you know another commutative operation?
If positive, what is it?

3. What does the word commutativity mean in general?

Suppose that we want to solve Problem 2 in the decimal place-value system. Thanks to commutativity, we can independently subtract single digits from the like, tens from tens, hundreds from hundreds, and so on. When needed, we can shift a digit to the previous position where it becomes ten units the position corresponds to. For example, one thousand becomes ten hundreds.

$$1 \times 10^3 = 10 \times 10^2$$

One hundred becomes ten tens.

$$1 \times 10^2 = 10 \times 10^1$$

Ten becomes ten ones.

$$1 \times 10^1 = 10 \times 10^0$$

This way,

$$\begin{array}{rcccc} & 2 & 0 & 1 & 2 \\ - & & 8 & 6 & 5 \\ \hline \end{array}$$

Problem 7 *Finish the above computation. Check if your solution to Problem 2 was correct.*

Let us now study one more important property of numbers, called *distributivity*. Distributivity is the following relation between multiplication and addition.

$$a \times (b + c) = a \times b + a \times c \qquad (1)$$

We already have used it above. Let us check it some more experimentally.

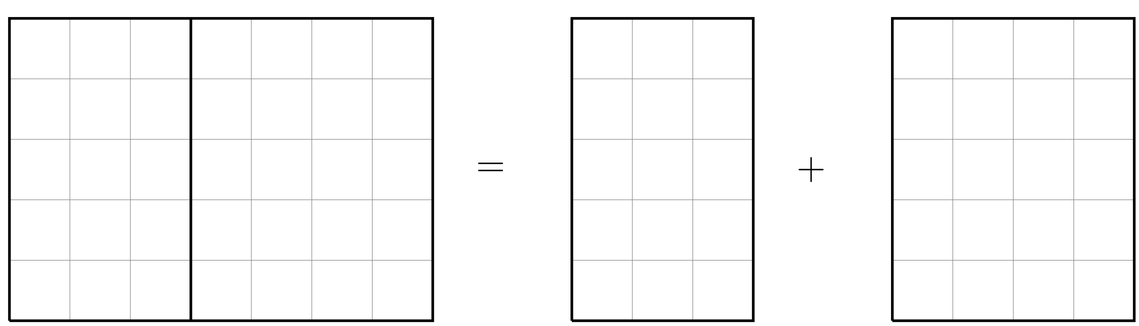
Problem 8

$5 \times (3 + 4) =$ $\qquad$ $5 \times 3 + 5 \times 4 =$

$7 \times 21 =$ $\qquad$ $7 \times 20 + 7 \times 1 =$

$49 \times 2 =$ $\qquad$ $40 \times 2 + 9 \times 2 =$

The first two exercises of Problem 8 have the following geometric interpretation.



One can similarly interpret the general law 1. If we have two gardens of the same width, a, and of lengths b and c respectively, then we can figure out the total area of the gardens in two ways.

We can either add up the lengths and consider the two gardens as one of width a and length $b+c$, or we can compute the areas $a \times b$ and $a \times c$ separately and then add them up. The result will obviously be the same.

That's how distributivity of multiplication helps to multiply big numbers written in the decimal place-value system.

Example 1

$$325 \times 67 = (3 \times 10^2 + 2 \times 10 + 5) \times (6 \times 10 + 7) =$$

$$3 \times 6 \times 10^{2+1} + 3 \times 7 \times 10^{2+0} + 2 \times 6 \times 10^{1+1} + 2 \times 7 \times 10+$$

$$5 \times 6 \times 10 + 5 \times 7 =$$

$$18,000 + 2,100 + 1,200 + 140 + 300 + 35 = 21,775$$

The example shows that all you need to know to multiply big numbers is the multiplication table for single digits! The rest is just shifting and adding.

Problem 9 *Use distributivity to compute* 84×136.

Let us redo the above problem using a slightly more efficient method called *long multiplication*. Let us write the factors one under the other so that single digits are written under single digits, tens under tens, hundreds under hundreds, and so on. Let us draw a line that will separate the input from the actual computation.

$$\begin{array}{rrrr} & & 8 & 4 \\ \times & 1 & 3 & 6 \\ \hline \end{array}$$

Let us first multiply the upper number by 6 and write the result under the line, single digits under single digits, tens under tens, etc.

$$\begin{array}{rrrr|l} & & 8 & 4 & \\ \times & 1 & 3 & 6 & \\ \hline & 5 & 0 & 4 & 84 \times 6 = 504 \end{array}$$

Let's do the same for 30, the next number of the decomposition $136 = 100 + 30 + 6$. The meaning of the number 30 is three tens, so to multiply by 30, we can multiply by three and write zero on the right of the result.

$$\begin{array}{rrrr|rcr} & & 8 & 4 & & & \\ \times & 1 & 3 & 6 & & & \\ \hline & 5 & 0 & 4 & 84 \times 6 & = & 504 \\ 2 & 5 & 2 & & 84 \times 30 & = & 2520 \end{array}$$

Multiplication by 100 simply shifts the first factor two positions to the left.

$$
\begin{array}{rrrr|rcr}
 & & 8 & 4 & & & \\
\times & 1 & 3 & 6 & & & \\
\hline
 & 5 & 0 & 4 & 84 \times 6 & = & 504 \\
2 & 5 & 2 & & 84 \times 30 & = & 2520 \\
8 & 4 & & & 84 \times 100 & = & 8400
\end{array}
$$

Since, thanks to distributivity, $84 \times 136 = 84 \times (100 + 30 + 6)$, all we need to finish the computation is to add up the lower three numbers on the left hand side.

$$
\begin{array}{rrrrr}
 & & & 8 & 4 \\
\times & & 1 & 3 & 6 \\
\hline
 & & 5 & 0 & 4 \\
+ & 2 & 5 & 2 & \\
 & 8 & 4 & & \\
\hline
1 & 1 & 4 & 2 & 4
\end{array}
$$

Problem 10 *Use long multiplication to compute the following.*

$$\begin{array}{r} 5\ 4 \\ \times\ 3\ 2 \\ \hline \end{array} \qquad \begin{array}{r} 4\ 5 \\ \times\ 2\ 3 \\ \hline \end{array}$$

$$\begin{array}{r} 1\ 2\ 7 \\ \times\ \quad\ 8 \\ \hline \end{array} \qquad \begin{array}{r} 3\ 4\ 6 \\ \times\ \ 5\ 2 \\ \hline \end{array}$$

The last property of numbers that we need to understand is the *associativity* of multiplication. We already know that multiplication is commutative, that is for any two numbers a and b it is always true that

$$a \times b = b \times a \tag{2}$$

Question 1 *Does the order of factors matter for more than two of them?*

To answer this question, let us first see if the following formula holds for any three numbers a, b, and c.

$$a \times (b \times c) = (a \times b) \times c \tag{3}$$

As always, it helps to experiment.

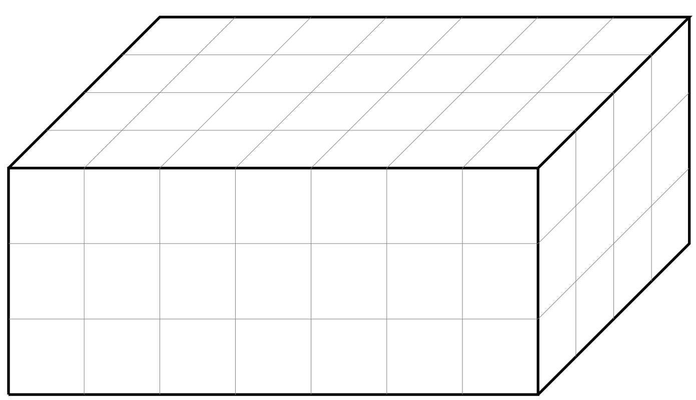
Problem 11 *Compute the following.*

$2 \times (5 \times 6) =$ $(2 \times 5) \times 6 =$

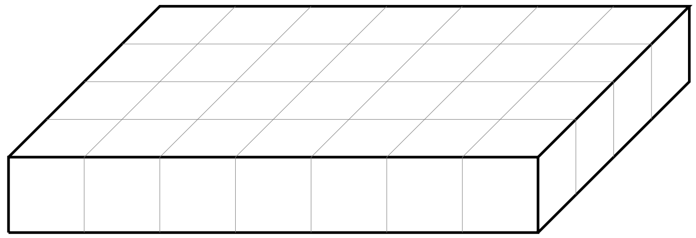
$3 \times (7 \times 4) =$ $(3 \times 7) \times 4 =$

$10 \times (8 \times 200) =$ $(10 \times 8) \times 200 =$

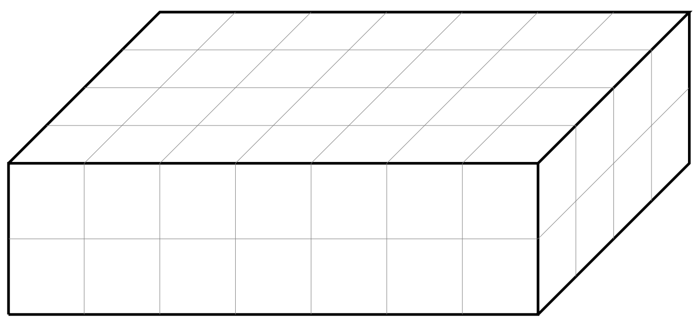
Volume is a natural way to measure 3D objects. It works the same way area works for 2D objects and length – for 1D. Consider the following rectangular prism made of cubes with the side length of 1'.



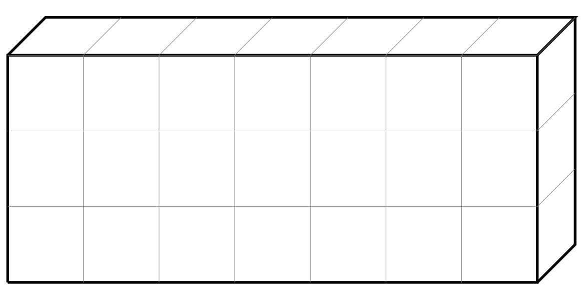
The volume of each of the cubes is one cubic foot. To figure out the volume of the prism, let us build it layer by layer. The ground layer is 7' long and 4' wide. It's volume is $7 \times 4 = 28$ cubic feet.



The middle layer has the same volume as the lower one, so the total volume of the first two layers is $28 + 28 = 2 \times 28 = 56$ cubic feet.

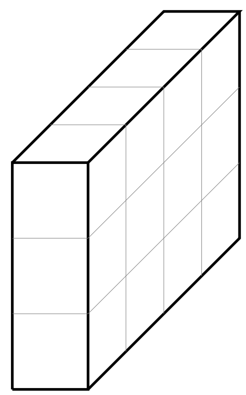


The prism is three layers tall, so the total volume is $3 \times 28 = 3 \times (7 \times 4) = 84$ cubic feet. However, we can construct the prism in a different way. Let us first build not the ground layer, but the front wall.



The volume of the wall is $3 \times 7 = 21$ cubic foot. The prism is made of four such layers, so its total volume is $(3 \times 7) \times 4 = 21 \times 4 = 84$ cubic feet. As we have checked in Homework Problem 11 by means of a direct computation, $3 \times (7 \times 4) = (3 \times 7) \times 4$.

Problem 12 *Suppose that the above prism is built in layers starting with its side wall.*



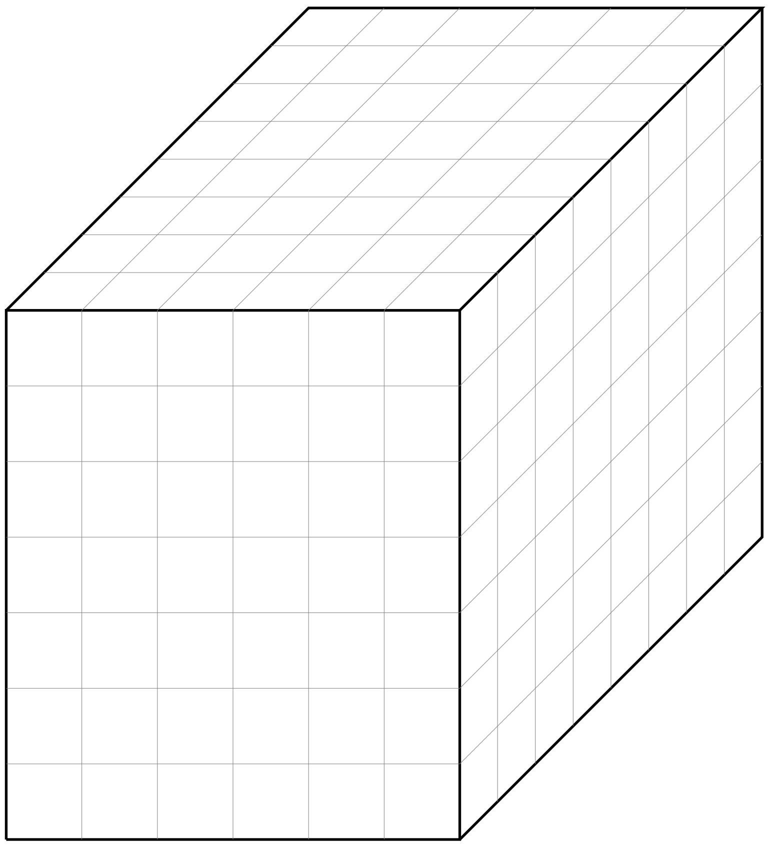
Write down the factors and put the parenthesis in the order that corresponds to figuring out the volume of the prism constructed this way.

The product of any three (positive real) numbers, a, b, and c, can be interpreted as figuring out the volume of a 3D rectangular prism a units long, b units wide, and c units tall. Similarly to the above consideration, we can compute the product any way we like, each of the ways corresponding to a different method of building the prism in layers.

Problem 13 *Does $a \times b \times c = c \times b \times a$ for all a, b, and c? Hint: employ commutativity of multiplication.*

Problem 14 *Is addition associative? In other words is $(a + b) + c$ always equal to $a + (b + c)$? Why or why not?*

Problem 15 *The side length of the below cubes is 1 mile. What is the volume of the prism made of them?*



Let us summarize. Numbers, whether written in a place-value or any other numeral system, have the following important properties.

1. Commutativity of addition and multiplication.

$$a + b = b + a \qquad a \times b = b \times a$$

2. Associativity of addition and multiplication.

$$a + (b + c) = (a + b) + c \qquad a \times (b \times c) = (a \times b) \times c$$

3. Distributivity of multiplication with respect to addition.

$$a \times (b + c) = a \times b + a \times c$$

The numeral system currently in use by humanity is place-value base ten. This means that we represent numbers as sums of powers of ten with coefficients taken from the set of ten digits,

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9.$$

This renders the following property. Multiplication of a number written in the decimal system by a positive integral power of ten, 10^n, shifts the number n positions to the left.