Binary Arithmetic and Truth Tables

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Adapted from worksheets by Oleg Gleizer

1 Binary Numbers

Let us recall that there are only two digits in the *binary system*, 0 and 1. $0_{10} = 0_2$ (remember, the subscript denotes the base), $1_{10} = 1_2$, but $2_{10} = 10_2$, $3_{10} = 11_2$, and so on.

Example 1. *Find the binary representation of the number* 174_{10}.

Let us list all the powers of 2 *that are less than or equal to* 174.

n	2^n
0	1
1	2
2	4
3	8
4	16

n	2^n
5	32
6	64
7	128
8	256

It turns out that the largest integral power of two still less than 174 *is* $128 = 2^7$.

$$174 = 128 + 46 = 2^7 + 46$$

The largest power of two less than 46 *is* $32 = 2^5$.

$$174 = 128 + 32 + 14 = 2^7 + 2^5 + 14$$

Finally, it is not hard to represent 14 *as a sum of powers of two,* $14 = 8 + 4 + 2$.

$$174 = 128 + 32 + 8 + 4 + 2 = 2^7 + 2^5 + 2^3 + 2^2 + 2^1$$

To write the number 174_{10} *in the binary form, we now need to fill the following eight boxes with either zeros or ones.*

7	6	5	4	3	2	1	0

The numbers under the boxes are the powers of two. If a power is absent from the decomposition of the number, then the corresponding box is filled with zero. For example, there is no $1 = 2^0$ *in the decomposition of the number* 174 *we have computed, so the first box from the right is filled with zero.*

							0
7	6	5	4	3	2	1	0

$2 = 2^1$ *is present in the decomposition, so the box corresponding to the first power gets filled with one.*

						1	0
7	6	5	4	3	2	1	0

$4 = 2^2$ *is also present in the decomposition, so the box corresponding to the second power of two gets filled with one.*

					1	1	0
7	6	5	4	3	2	1	0

Filling up all the boxes gives us the binary representation.

1	0	1	0	1	1	1	0
7	6	5	4	3	2	1	0

We write it down as follows.

$$174_{10} = 10101110_2$$

Problem 1. *Find the binary representations of the following decimal numbers.*

$$12_{10} =$$

$$25_{10} =$$

$$32_{10} =$$

$$100_{10} =$$

Problem 2. *Find the decimal representations of the following binary numbers.*

$$101_2 =$$

$$11001_2 =$$

$$1000000_2 =$$

$$1010011_2 =$$

Problem 3. *Use long addition to sum up the following two binary numbers without switching to the decimals.*

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 1 \\ +\quad 1\ 0\ 0\ 1\ 1 \\ \hline \end{array}$$

Then find the decimal representation of the summands and of the sum and check your answer.

$$110111_2 =$$

$$10011_2 =$$

Problem 4. *Perform the following subtraction of the binary numbers.*

$$\begin{array}{r} 1\ 0\ 0\ 1\ 0 \\ -\quad 1\ 0\ 1\ 1 \\ \hline \end{array}$$

Then find the decimal representations of the numbers and of the difference and check your answer.

$10010_2 =$

$1011_2 =$

Problem 5. *Perform the following long multiplication without switching to the decimals.*

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1 \\ \times \quad 1\ 0\ 1\ 0 \\ \hline \end{array}$$

Then find the decimal representations of the factors and of the product and check your answer.

$$11011_2 =$$

$$1010_2 =$$

Problem 6. *Solve the following equations in the binaries.*

$$x + 11 = 1101 \qquad\qquad x =$$

$$x - 10 = 101 \qquad\qquad x =$$

$$x - 1101 = 11011 \qquad\qquad x =$$

$$x + 1110 = 10001 \qquad\qquad x =$$

$$x + 111 = 11110 \qquad\qquad x =$$

$$x - 1001 = 11101 \qquad\qquad x =$$

2 Statements and Truth Tables

A *statement* is an expression which is either *True* or *False*. For example, "Let's go!" is not a statement, while "My math teacher is not human!" is.

Problem 7. *In the space below, write two sentences that are statements in the above sense and two more that are not.*

-
-
-
-

If a statement A is true, we write $A = 1$. If a statement A is false, we write $A = 0$.

Problem 8. *Determine which of the sentences below are statements and find their values.*

A 23 *is divisible by* 5.

B *Please don't smoke on board the aircraft.*

C $7x + 5y = 70$

D *Pyotr Tchaikovsky is a famous Russian hockey player.*

E *What time is it now?*

F *Get out of here!*

G *Math is fundamental for understanding all other sciences.*

If a statement mentions only one event, true or false, it is called *simple*. If a statement mentions more than one event, it is called *composite*. For example, the statement *I come to the Math Circle by car* is simple, while the statement *I come to the Math Circle by car or by bus* is composite.

Let A and B be statements. Let us define $A+B$ as the statement A *or* B. For example, if $A =$ *three is greater than two* and $B =$ *three is greater than five*, then $A + B =$ *three is greater than two or than five.*

The statement A *or* B is false if and only if both A and B are false. If either of the statements A or B is true, then A *or* B is true as well.

A	B	$A+B$
0	0	0
1	0	1
0	1	1
1	1	1

We can see from the above *truth table* that in the algebra of logic $0 + 0 = 0$, while $1 + 0 = 0 + 1 = 1 + 1 = 1$.

Problem 9. *Is the logical addition commutative? Why or why not? Please write down your explanation in the space below.*

Problem 10. *Prove that* $A + 0 = A$ *and* $A + 1 = 1$.

Give a verbal interpretation to the above algebraic statements.

Problem 11. *Prove that* $\underbrace{A + A + \ldots + A}_{n \text{ times}} = A$.

Problem 12. *Form the logical sum of the following three statements and find its value.*

$A =$ *The planet of Earth rotates around the North Star.*

$B =$ *The planet of Earth rotates around Alpha Centauri.*

$C =$ *The planet of Earth rotates around the Sun.*

$A + B + C =$

Problem 13. *Prove that for the logical addition,* $(A + B) + C = A + (B + C)$. *Hint: use the truth table below.*

A	B	C	$A+B$	$B+C$	$(A+B)+C$	$A+(B+C)$
0	0	0				
1	0	0				
0	1	0				
0	0	1				
1	1	0				
1	0	1				
0	1	1				
1	1	1				