ORMC Olympiad Group
Week 1

Osman Akar

October 2, 2021

0.1 Bernoulli Polynomials and Summation Formulas

Utilizing a cleaver method to add up a sequence of numbers is one of the most naive usage of mathematics. We will begin the first lecture with the summation formulas that the reader probably already know.

- $S_n^1 := 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$[1]
- $S_n^2 := 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$
- $S_n^3 := 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$[2]

These formulas have a number of proofs. The first one can be proved using only figures[3]. Another way is to recognize $1+n = 2+(n-1) = 3+(n-2) = \cdots = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil$, which allows us to couple numbers $(1, n), (2, n-1), \ldots$ then add them up, which gives the result $S_n^1 = (n+1)\frac{n}{2}$. This is the idea that Gauss figured out when he was in elementary school[4].

Inductive proofs works for all. Indeed, assuming $S_n^1 = \frac{n(n+1)}{2}$, you can easily proof $S_{n+1}^1 = S_n^1 + n + 1 = \frac{n(n+1)}{2} + n + 1 = \frac{(n+1)(n+2)}{2}$. The same method works for S_n^2 and S_n^3 as well.

Having these formulas, a natural question arises: Do we have a formula for $S_n^k := 1^k + 2^k + \cdots + n^k$ in terms of n and k? And if so, what is it and how can we find it? The answer is yes (kind of)! We do not have an elementary precise formula involving n and k[5], but we can find the formula in terms of n for any given k. To do this, we use something called *Bernoulli Polynomials*.

Definition 0.1.1. Bernoulli Polynomials A polynomial p with degree k

[1] The reader sees := instead of only =. We will use := when we first define a new variable or function.

[2] Realize here a wonderful equality $S_n^3 = (S_n^1)^2$

[3] If we put columns length $n, n-1, \ldots, 1$ side by side (assuming each column is divided by identical bricks), we get a stair of length n. There are in total $n+(n-1)+\cdots+1 = S_n^1$ bricks inside a stair. If we place another stair with the same size above as in the figure, and fill the main diagonal with $n+1$ bricks, we get a square with size $n+1$. There are in total $2S_n^1 + (n+1) = (n+1)^2$ bricks, which proves $S_n^1 = \frac{(n+1)^2-(n+1)}{2} = \frac{n(n+1)}{2}$.

[4] Click here to read full story.

[5] Although there is *Faulhaber's formula* which involves *Bernoulli Numbers*, but at this moment they are advanced

is a special function from reals to reals

$$p(x) = a_k x^k + a_{k-1} x^{k-1} + \dots a_1 x + a_0$$

where $a_i \in \mathbb{R}$ for all i. k^{th} degree *Bernoulli Polynomial* is a special polynomial that satisfies

$$B^k(x) - B^k(x-1) = x^{k-1}$$

Why *Bernoulli Polynomials* are important to find the sum S_n^k? It is simply because we can write S_n^n as a telescopic sum of bernoulli polynomials

$$S_n^k = \sum_{i=1}^{n} i^k = \sum_{i=0}^{n} B^{k+1}(i) - B^{k+1}(i-1) = B^{k+1}(n) - B^{k+1}(0)$$

So knowing k^{th} degree *Bernoulli Polynomial* we can find a formula for S_n^k. Therefore we see that S_n^k is actually a $k+1$ degree polynomial of n. E.g., for $B^2(x) = \frac{1}{2}x^2 + \frac{1}{2}x = \frac{x(x+1)}{2}$, we have $B^2(x) - B^2(x-1) = \frac{x(x+1)}{2} - \frac{(x-1)x}{2} = x$, so $S_n^1 = B^2(n) - B^2(0) = \frac{n(n+1)}{2}$.

But how do we find *Bernoulli Polynomials* without guess and checking?

Example 0.1.1. Find third degree Bernoulli Polynomial $B^3(x)$ and confirm the formula of S_n^2.

Solution Write $B^3(x) = ax^3 + bx^2 + cx + d$. Our goal is to find coefficients a, b, c, d so that $B^3(x) - B^3(x-1) = x^2$. Note that d cancels out, so we may assume $d = 0$. Then

$$\begin{aligned} x^2 &= B^3(x) - B^3(x-1) \\ &= ax^3 + bx^2 + cx - a(x-1)^3 - b(x-1)^2 - c(x-1) \\ &= ax^3 + bx^2 + cx - a(x^3 - 3x^2 + 3x - 1) - b(x^2 - 2x + 1) - c(x-1) \\ &= x^2(3a) + x(-3a + 2b) + a - b + c \end{aligned}$$

Thus to have the equality, we need to have $3a = 1, -3a+2b = 0, a-b+c = 0$, which gives $a = \frac{1}{3}, b = \frac{1}{2}, c = \frac{1}{6}$. Then $B^3(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x = \frac{2x^3+3x^2+x}{6} =$

$\frac{x(x+1)(2x+1)}{6}$. Finally

$$S_n^2 = \sum_{i=1}^{n} i^2 = \sum_{i=0}^{n} B^3(i) - B^3(i-1) = B^3(n) - B^3(0) = \frac{n(n+1)(2n+1)}{6}$$

0.2 Problems

1. Using Bernoulli polynomials, prove $S_n^3 = \frac{n^2(n+1)^2}{4}$ and find formula for $S_n^4 = 1^4 + 2^4 + 3^4 + \cdots + n^4$

2. **(AMC12 2000)** Figures 0, 1, 2, and 3 consist of 1, 5, 13, and 25 nonoverlapping unit squares, respectively. If the pattern were continued, how many nonoverlapping unit squares would there be in figure 100?

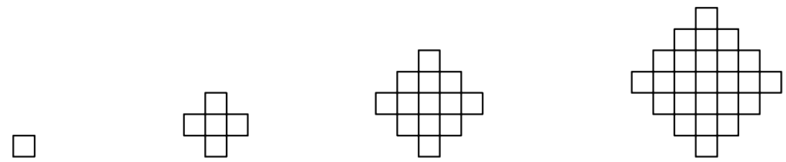

Figure 0   Figure 1   Figure 2   Figure 3

3. Consider the sequence $1, 7, 19, 37, 61, \ldots$.
   - (i) What is the next term?
   - (ii) What is the recursive relationship between the terms?
   - (iii) Starting $a_1 = 1, a_2 = 7, \ldots$, find general formula for a_n. The formula should only consist of the term n.
   - (iv) How many positive integer divisors does $a_1 + a_2 + a_3 + \ldots + a_{100}$ have?

4. Compute $3 \cdot 5 + 4 \cdot 7 + 5 \cdot 9 + 6 \cdot 11 + \cdots + 13 \cdot 25$.

5. (a) Compute the sum $1^2 + 3^2 + 5^2 + \cdots + 49^2$ without using calculator.
   (b) You know the formula for $1^2 + 2^2 + 3^2 \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$. Using this, create a formula for $1^2 + 3^2 + 5^2 \cdots + (2n-1)^2$

6. Find the mean of the numbers $1,\ -2^2,\ 3^2,\ -4^2, \ldots, -100^2,\ 101^2$.

7. **(TJNMO-FR 2009)** Find the sum of irreducible fractions of the form $\frac{n}{5}$ between 12 and 21.

8. Find formula for $S_n^5 = 1^5 + 2^5 + 3^5 + \cdots + n^5$

9. Find the remainder when $1^2 + 4^2 + 7^2 + 10^2 + \cdots + 97^2 + 100^2$ divided by 1000.

10. Consider the sequence $1, 2, 5, 12, 27, 58, 121, ...$

    (i) What is the next term?

    (ii) What is the relation between the terms and index?

    (iii) Find general formula for a_n.

    (iv) What is the sum $a_1 + a_2 + ... + a_{99}$?

    (v) Find general formula for the sum $a_1 + a_2 + ... + a_n$.