Nimbers

Aaron Anderson *

October 3, 2021

1 Impartial Games

Definition 1. A *combinatorial game* is a game that is

- two-player (two players alternate turns)
- complete-information (no face-down cards)
- deterministic (no randomness)
- well-founded (it's guaranteed to finish in a finite amount of turns)
- where the first player who cannot move loses, and thus the other player wins

We will start by studying *impartial* combinatorial games, which are games where the rules are the same for each player. It may still be better to be the player who goes first or second.

Problem 1. Here are some familiar examples of games. Do these meet the definitions, and are they impartial?

- Chess
- Checkers
- Poker
- Football (whatever that means to you)

When we study games, we will think about *positions* - a position is the state of the game, recording the possible moves for each player, right before a particular turn.

In an impartial combinatorial game, a position is called an *N-position* if the *N*ext player to move is guaranteed to win, or a *P-position* if the *P*revious player is guaranteed to win.

Problem 2. Prove that each each position in an impartial combinatorial game is guaranteed to either have the first or second player win - and is thus either an N-position or a P-position. (Hint: If you have a game where every move has a winning strategy, show that the game itself has a winning strategy. Then if you have a game without a winning strategy, find a contradiction to well-foundedness.)

*with thanks to John Conway and Alfonso Gracia-Saz

1.1 Nim

Perhaps the most classic impartial game to study is Nim. There are a few piles of objects (for our purposes, a few different groups of party parrots on a Jamboard, but I recommend beans or coins if you play this IRL). On each turn, you take a positive number of parrots/beans/coins from *exactly one* pile. Play proceeds until someone empties the last pile, and thus wins, because there is no valid move for the other player.

Problem 3. I've prepared a jamboard with some Nim games, have fun! If your group has some people who have not played Nim before, they should try to play each other first, and then face more experienced players.

Play Jamboard slides 1 - 3.

Problem 4. Say we're playing Nim with two piles - with m and n parrots respectively (m and n could be zero, in which case there are really 0 or 1 piles). For what values of m and n is this a P-position, or an N-position?

Problem 5. Say you have an impartial game, and an algorithm for determining whether any given position is a P- or N-position. Describe a winning strategy for the game, assuming you get to choose whether you go first or second at the beginning.

1.2 Other Subtraction Games

Subtraction games are variations on Nim - still impartial games. Let S be a set of positive integers. Then once again let us populate some piles with parrots, and once again you remove them from one pile at a time, but now you are only allowed to move a number of parrots in S.

Problem 6. Try Jamboard slides 4 through 6 - each will specify a set S. Feel free to mix and match S with the configuration, and generally play games as much as you want until you get a feel for the strategy.

1.3 Adding Games Together

I want to teach you a new board game. It's called *Chess + Checkers*. To play, we sit down on opposite sides of a table, and set up a chessboard and a checkerboard side-by-side. On each of our turns, we pick one of the boards and do a move in that board. As if chess and checkers aren't hard enough (well, maybe checkers isn't hard enough), now you have to decide whether to respond to my threat in chess, or get ahead in checkers.

In general, we can add any two combinatorial games G_1, G_2. In the new game, $G_1 + G_2$, each player gets to move in only one game at a time, and the person who runs out of moves in *both* games loses. (That is, if G_1 is finished first, no matter who won G_1, the winner of G_2 is the overall winner.)

Problem 7. Convince yourself that the sum of two impartial games is impartial, and the sum of two games of Nim is also game of Nim (how many piles are there? of what sizes?).

Problem 8. Is this addition commutative? Is it associative? (Reminder: Commutativity means that $a+b=b+a$ for all a, b. Associativity means that for all a, b, c, we have $(a+b)+c = a+(b+c)$.)

Problem 9. Let G be an impartial game. Who wins $G + G$?

Definition 2. We say that two games G_1, G_2 are *equivalent* (written $G_1 \approx G_2$) if for every game H, the sum $G_1 + H$ is a second-player win (P-position) if and only if the sum $G_2 + H$ is.

Problem 10. Show that this notion of equivalence is an *equivalence relation* by checking these three statements:

- For any game G, $G \approx G$
- For any games G_1, G_2, $G_1 \approx G_2$ if and only if $G_2 \approx G_1$
- For any games G_1, G_2, G_3, if $G_1 \approx G_2$ and $G_2 \approx G_3$, then $G_1 \approx G_3$.

Problem 11. Show that if $G_1 \approx G_2$ and $H_1 \approx H_2$, then $G_1 + H_1 \approx G_2 + H_2$.

Definition 3. Let 0 be the game with no valid moves for either player (for instance, Nim with 0 piles).

Problem 12. Show that for any game G, $G + 0 = G$.

Problem 13. Show that a game G is equivalent to 0 if and only if G starts in a P-position.

Problem 14. Let G_1, G_2 be impartial games. Prove that $G_1 \approx G_2$ if and only if $G_1 + G_2 \approx 0$.

1.4 Nimbers

Definition 4. Let G be an impartial game with only finitely many valid moves at each position (like, for instance, Nim with finitely many parrots).

Define the *Sprague-Grundy* value of G as follows:

First calculate the Sprague-Grundy value of each position you can legally move to. Then let the Sprague-Grundy value be the smallest natural number that isn't on that (finite) list.

Problem 15. What is the Sprague-Grundy value of the empty game (with no valid moves)?

Problem 16. For a natural number n, let $*n$ represent a Nim game with one pile of n parrots (Here $*0 = 0$). What is the Sprague-Grundy value of $*n$?

Problem 17. Prove that any impartial game with Sprague-Grundy value n is equivalent to $*n$. (Hint: Assume this is true for every position you can move to, and do a kind of induction.)

This is actually pretty remarkable. Any finite-move impartial game is equivalent to a single pile of Nim! This even gives us a winning strategy! For finite-move impartial games, we can recursively determine any position's Sprague-Grundy value, and then we know whether that position is a P-position or an N-position! Then it's just a matter of trying to move to P-positions whenever possible.

Problem 18. Let's try a different Nim-ish game. There is one pile of n parrots, and at each turn, you must remove a number of parrots which is a power of 4. What is the Sprague-Grundy value of this game for each n? For which values of n is this a P- or N-position?

Problem 19. Let's go back to two-pile Nim, which we now refer to as $*m + *n$. What's the Sprague-Grundy value?

Using this, describe a strategy for general Nim, that is, $*n_1 + *n_2 + \cdots + *n_k$.

We've now developed the basic theory of nimbers. A *nimber* is just an impartial game, where two nimbers are considered the same if the games are equivalent. We've proven (at least in the finite-move case) the *Sprague-Grundy Theorem*, which says that each impartial game is equivalent to a single-pile version of Nim, so each nimber can be represented as a single-pile version of Nim, hence the name. These have a special addition, an additive identity, and in fact they follow (most of) the usual axioms of addition.

Problem 20. For what values of n is the set $\{*0, *1, *2, \ldots, *(n-1)\}$ closed under addition? (That is, if $i, j < n$, then $*i + *j = *k$ for some $k < n$.)

2 Multiplying Nimbers

We'll now define another game that will allow us to extend our arithmetic on nimbers.[1] We play on a grid, where we place counters/parrots on lattice points of the grid. As before, this will be an impartial game, where the players take turns playing legal moves until someone makes the last possible legal move and thus wins. To make a move, pick a counter and do one of the following:

- Remove it
- Move it to the left (as many spaces as you want)
- Move it down (as many spaces as you want)
- If this counter is at the upper-right corner of some rectangle, remove this counter and add counters to the other three corners

If after this, you end up with two counters on the same grid point, remove both of them.

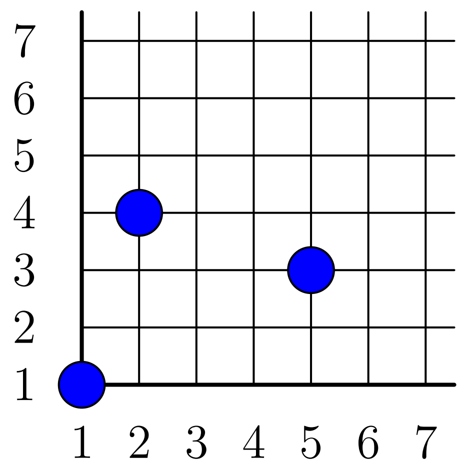

Figure 1: A valid setup with 3 counters

Problem 21. Now try playing the games on Jamboard slides 7-9.

We will find the strategy soon, but think about the following questions: Who will win each game, and is it possible to change the winner by adding a single counter on the diagonal?

Problem 22. Let's say we have a configuration of this game where all the counters are in the leftmost column (or equivalently the bottom row). Explain how to calculate its Sprague-Grundy value. What does this have to with Nim?

Problem 23. Explain how if you added another row and column (each labelled 0), you could summarize all moves as just the rectangle rule, with extra step that at the end of each move, you remove all counters on row 0 or column 0.

Definition 5. If $m, n \in \mathbb{N}$, we can also define multiplication on nimbers: $*m \cdot *n$ is the unique Nim game equivalent to the game $\{*m \cdot *n' + *m' \cdot *n + *m' \cdot *n' : m' < m, n' < n\}$.

This multiplication is associative ($*k \cdot (*m \cdot *n) = (*k \cdot *m) \cdot *n$) and distributes with addition ($*k \cdot (*m + *n) = *k \cdot *m + *k \cdot *n$), but we will not prove this.

[1] This game comes from this blog, and is based on a game by Lenstra.

Problem 24. If $m, n \in \mathbb{N}$, place a counter at the position (m, n) on the grid in our new game. Show that the S-G value of this game is $*m \cdot *n$.

Problem 25. Check that for all n, $*0 \cdot *n = *0$ and $*1 \cdot *n = *n$.

Problem 26. Calculate a multiplication table for the nimbers $\{*0, *1, *2, *3\}$.

Problem 27. Prove that for all $a, b > 0$, $*a \cdot *b \neq *0$.

Problem 28. It turns out that for all $m \neq n$, $*2^{2^m} \cdot *2^{2^n} = *2^{2^m+2^n}$, and $*2^{2^n} \cdot *2^{2^n} = *(3 \cdot 2^{2^n-1})$.

- Use these facts (and distributivity and such) to calculate $*8 \cdot *8$.
- Explain how you can use these rules to calculate Nim-multiplication in general.
- Show that the set $\{*0, *1, \ldots, *(2^{2^N} - 1)\}$ is closed under multiplication.

Problem 29. Assuming that $\{*0, *1, *2, \ldots, *(2^{2^N} - 1)\}$ is closed under multiplication, prove that for all $0 < n < 2^{2^N}$, there exists $m < 2^{2^N}$ such that $*m \cdot *n = *1$.

Problem 30. Prove that any arrangement in this game can be turned into a second-player win by adding or removing a counter from the diagonal.

(Hint: what does this mean in algebraic terms? Show also that every position with only a single counter on the diagonal has a different value.)

Problem 31. Prove the assertions we've made so far by induction. Specifically, show the following by induction on N:

- If $m < n < N$, then $*2^{2^m} \cdot *2^{2^n} = *2^{2^m+2^n}$.
- If $n < N$, then $*2^{2^n} \cdot *2^{2^n} = *(3 \cdot 2^{2^n-1})$.
- The set $\{*0, *1, \ldots, *(2^{2^N} - 1)\}$ is closed under multiplication.