ORMC Intermediate Week 1

Kevin Li, Yan Tao

October 3, 2021

1 Introduction

Hello everyone! Welcome to week 1 of Math Circle. Today, we will be working in small groups of about 5-6 people and working on some problems in different areas of math. There are 4 main areas for today's worksheet: **Algebra, Arithmetic, Geometry,** and **Combinatorics + Probability**. We will aim to spend about 20-25 minutes on each section. Don't worry if you are unable to solve some of the problems, just do your best! It would be preferred for everyone to show as much work as they think necessary to be able to explain their solution clearly. There will be time to discuss your solutions with one another. If you finish early with each section, there is one challenge problem at the end of the worksheet related to each topic area of focus today. Have fun!

2 Problems

2.1 Algebra Problems

Problem 1. What is the value of the expression $\sqrt{16\sqrt{8\sqrt{4}}}$?

Problem 2. Qiang drives 15 miles at an average speed of 30 miles per hour. How many additional miles will he have to drive at 55 miles per hour to average 50 miles per hour for the entire trip?

Problem 3. The harmonic mean of a set of non-zero numbers is the reciprocal of the average of the reciprocals of the numbers. What is the harmonic mean of 1,2, and 4? (Remember, the reciprocal of a number, n, is $1/n$)

Problem 4. Chloe and Zoe are both students in Ms. Demeanor's math class. Last night they each solved half of the problems in their homework assignment alone and then solved the other half together. Chloe had correct answers to only 80% of the problems she solved alone, but overall 88% of her answers were correct. Zoe had correct answers to 90% of the problems she solved alone. What was Zoe's overall percentage of correct answers?

Problem 5. Starting with some gold coins and some empty treasure chests, I tried to put 9 gold coins in each treasure chest, but that left 2 treasure chests empty. So instead I put 6 gold coins in each treasure chest, but then I had 3 gold coins left over. How many gold coins did I have?

2.2 Arithmetic Problems

Problem 6. How many different real numbers x satisfy the equation $(x^2 - 5)^2 = 16$?

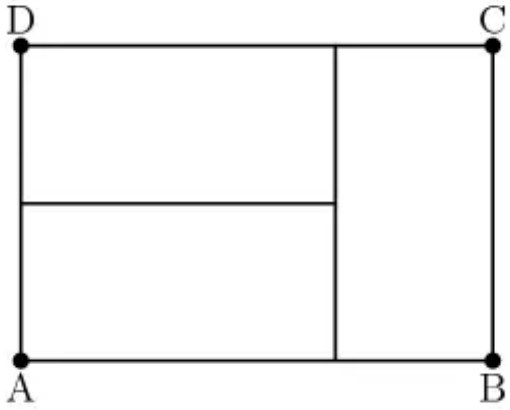
Problem 7. What is the median and mean of the set $\{3, 6, 9, 10\}$? A fifth number, n, is added to the set $\{3, 6, 9, 10\}$ to make the mean of the set of five numbers equal to its median. The number of possible values of n is

Problem 8. The product of the two 99-digit numbers $303,030,303,...,030,303$ and $505,050,505,...,050,505$ has thousands digit A and units digit B. What is the sum of A and B? For example, the number 123,456,789 has thousands digit 6 and units digit 9.

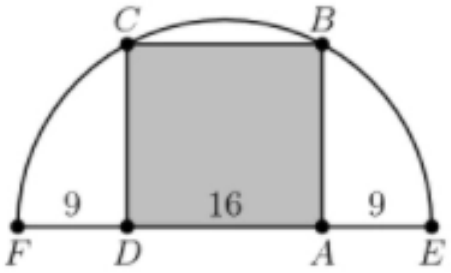
Problem 9. How many positive cubes divide $3! \cdot 5! \cdot 7!$?

2.3 Geometry Problems

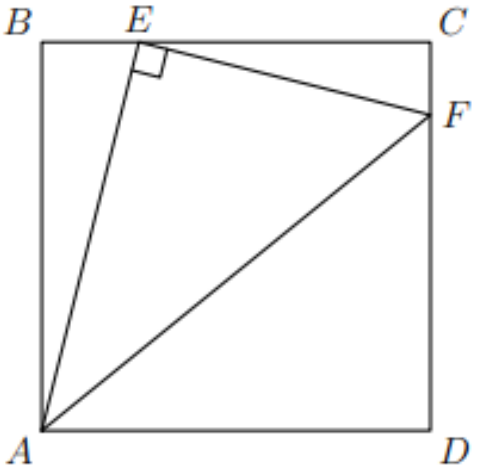
Problem 10. Three identical rectangles are put together to form rectangle $ABCD$. Given that the length of the shorter side of each rectangle is 5 feet, what is the area of rectangle $ABCD$ in square feet?



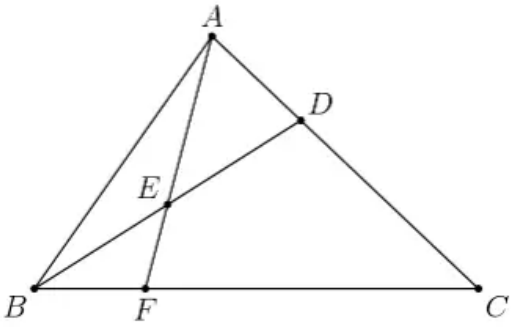
Problem 11. Rectangle $ABCD$ is inscribed in a semicircle with diameter $\overline{FE}$ as shown in the figure. Suppose $DA = 16$ and $FD = AE = 9$. What is the area of $ABCD$?



Problem 12. Triangle AEF is a right triangle with $AE = 4$ and $EF = 3$. The triangle is inscribed inside square $ABCD$ as shown. What is the area of the square?



Problem 13. In triangle ABC, point D divides side $\overline{AC}$ so that $AD : DC = 1 : 2$. Let E be the midpoint of $\overline{BD}$ and let F be the point of intersection of line BC and line AE. Given that the area of triangle ABC is 360, what is the area of triangle EBF?



2.4 Combinatorics and Probability Problems

Problem 14. Alice has 24 apples. In how many ways can she share them with Becky and Chris so that each of the people has at least 2 apples?

Problem 15. From a regular octagon, a triangle is formed by connecting three randomly chosen vertices of the octagon. What is the probability that at least one of the sides of the triangle is also a side of the octagon?

Problem 16. The faces of each of two fair dice are numbered 1, 2, 3, 5, 7, and 8. When the two dice are tossed, what is the probability that their sum will be an even number?

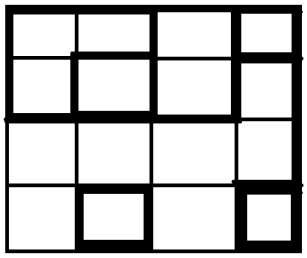
Problem 17. Ten balls numbered 1 to 10 are in a jar. Jack reaches into the jar and randomly removes one of the balls. Then Jill reaches into the jar and randomly removes a different ball. What is the probability that the sum of the two numbers on the balls removed is even?

Problem 18. How many 7-digit palindromes (numbers that read the same forwards and backwards) that do not have more than 2 occurrences of any digit are there?

3 Challenge Problems

Problem 19. The polynomial $P(x) = x^3 + ax^2 + bx + c$ has the property that the mean of its zeros, the product of its zeros, and the sum of its coefficients are all equal. If the y-intercept of the graph of $y = P(x)$ is 2, what is b?

Problem 20. Complete the number puzzle below. Clues are given for the four rows. (Answers may not begin with a zero.) One cell is already provided. Cells inside a region must all contain the same digit, and each region contains a different digit. (Regions are denoted by bold lines)



1. Row 1's digits backwards form a multiple of 64.
2. Row 2's digits from a decreasing arithmetic progression.
3. Row 3 is the product of exactly three prime numbers.
4. Row 4 can be written as $(n-2)(n+2)$ for some integer n.

Problem 21. Four distinct points are arranged in the plane so that the segments connecting them have lengths $a, a, a, a, 2a$, and b. What is the ratio of b to a?

Problem 22. A teacher has 5 students, all of whom have hats. The students exit the classroom and leave their hats, and the instructor permutes the students' hats. How many ways are there for the instructor to do this such that *exactly* one of the students keeps their own hat?