Summer Session Week 7

Solutions

August 2021

Exercise 1: After drawing the Venn diagram, we see that we can write $A = (A \setminus B) \cup B$, where this union is disjoint. Thus by theorem, $|A| = |A \setminus B| + |B|$, so $|A| - |B| = |A \setminus B|$.
Exercise 2: Looking at the Venn diagram again, we see $A \cup B = (A \setminus (A \cap B)) \cup (A \cap B) \cup (B \setminus (A \cap B))$, where this union is disjoint. Thus by theorem and induction, we have $|A \cup B| = |A \setminus (A \cap B)| + |A \cap B| + |B \setminus (A \cap B)| = |A| - |A \cap B| + |A \cap B| + |B| - |A \cap B| = |A| + |B| - |A \cap B|$. Note $|A \cap B| - |A \cap B| = 0$ as $|A \cap B| = 0 + |A \cap B|$. This does not contradict theorem as we are not assuming $A \cap B = \emptyset$.
***Exercise 3**: Write the union as a disjoint union: $A \cup B \cup C = (A \setminus ((A \cap B) \cup (A \cap C))) \cup (B \setminus ((A \cap B) \cup (B \cap C))) \cup (C \setminus ((A \cap C) \cup (B \cap C))) \cup ((A \cap B) \setminus (A \cap B \cap C)) \cup ((A \cap C) \setminus (A \cap B \cap C)) \cup ((B \cap C) \setminus (A \cap B \cap C)) \cup (A \cap B \cap C)$, so by theorem and a little rearranging we get $|A \cup B \cup C| = |A| + |B| + |C| + |A \cap B| + |A \cap C| + |B \cap C| + |A \cap B \cap C| - 3|A \cap B \cap C| - |(A \cap C) \cup (B \cap C)| - |(A \cap B) \cup (B \cap C)| - |(A \cap B) \cup (A \cap C)|$. After applying exercise 2 to the last three terms and canceling, we get the answer.
Exercise 4: The function $f : A \to \mathcal{P}(A), f(a) = \{a\}$ is an injection.
***Exercise 5**: Assume for contradiction that there exists a bijective map from A onto $\mathcal{P}(A)$. As in the hint, make the set $R = \{a \in A | a \notin f(a)\}$. This makes sense as $f(a)$ is a subset of A, and $a \in A$, so we can ask whether or not a is in $f(a)$. Now, clearly $R \in \mathcal{P}(A)$ as it is a subset of A. As f is surjective, there must be an $a \in A$ such that $f(a) = R$. If $a \in R$, then $a \notin f(a)$, so $f(a) \neq R$ as a is in one but not the other. So $a \notin R$. But then $a \in f(a)$. Again, $f(a) \neq R$. This is a contradiction to the existence of a, so f cannot exist.
Exercise 6: This is week 4, exercise 14 in disguise.
Exercise 7: This is week 4, exercise 15 in disguise.
Exercise 8: The function $f(x) = x$ is an injection from any set to itself.
Exercise 9: This is week 4, exercise 16 (Schröder-Bernstein Theorem) in

disguise, along with this week's exercise 7.
Exercise 10: If $f : A \to B$ and $g : B \to C$ are injections, then $g \circ f : A \to C$ is an injection by week 6 exercise 16.
Exercise 11: The map $f(x) = x$ is a bijection between any set and itself, so $a = a$ for any a. Thus a cannot be strictly less than a.
Exercise 12: Suppose $a < b$. For contradiction, assume $b < a$. Then by Schröder-Bernstein Theorem, $a = b$. But this contradicts the fact that $a < b$. So we do not have $b < a$.
Exercise 13: We already have that $a \leq c$. Suppose $a = c$; then there is a bijection $f : A \to C$, where $|A| = a$ and $|C| = c$. As $b < c$, there is a surjection g from C to B. If we compose f with g, we get a surjection $f \circ g : A \to B$, which tells us that $a > b$. But this contradicts the previous exercise, as we assumed $a < b$. Thus $a \neq c$ and $a \leq c$, so $a < c$.
Exercise 14: The map $f(x) = x$ is a bijection between any set with itself.
Exercise 15: Suppose $f : A \to B$ is a bijection. Then the map $f^{-1} : B \to A$ is also a bijection.
Exercise 16: If $f : A \to B$ and $g : B \to C$ are bijections, then $g \circ f : A \to C$ is a bijection by week 6 exercise 18.
Exercise 17: We are given that there are bijections $f : A \to B$ and $g : C \to D$. Define $\phi : A \times C \to B \times D$, where $\phi(a, c) = (f(a), g(c))$. It is not hard to show that ϕ is a bijection.
Exercise 18: See week 4 exercise 13.
Exercise 19: $f(x) = (b - a)x + a$
Exercise 20: Let $a_1, a_2, \ldots$ be a countable subset of A. Define $f : A \to A \cup \{a\}$ by

$$f(x) = \begin{cases} x & \text{if } x \neq a_1, a_2, \ldots \\ a_{n-1} & \text{if } x = a_2, a_3, \ldots \\ a & \text{if } x = a_1 \end{cases}$$

Exercise 21: Use exercise 20 with $A = (0, 1)$ and $a = \pm 1$.
Exercise 22: The function $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is a bijection between $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $\mathbb{R}$. Then use exercise 19.
Exercise 23: The map $f : (0, 1) \to (1, \infty), f(x) = \frac{1}{x}$ is a bijection. Then apply previous exercises.
***Exercise 24**: The map $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}, f(n) = (n, 0)$ is an injection, so $|\mathbb{N}| \leq |\mathbb{N} \times \mathbb{N}|$. Using the hint, we get an injection the other direction. It is indeed an injection as if $f(m, n) = f(m^*, n^*)$, then $2^m \cdot 3^n = 2^{m^*} \cdot 3^{n^*}$. By the uniqueness of prime factorization, we must then have $m = m^*$ and $n = n^*$. Thus $|\mathbb{N} \times \mathbb{N}| \leq |\mathbb{N}|$, so by Schröder-Bernstein, $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$.
***Exercise 25**: The map $f : \mathbb{N} \to \mathbb{Q}, f(n) = n$, is an injection, so $|\mathbb{N}| \leq |\mathbb{Q}|$. By the hint, we get a surjection from $\mathbb{N}$ onto $\mathbb{Q}$, so $|\mathbb{N}| \geq |\mathbb{Q}|$. Thus by

exercise 9, $|\mathbb{N}| = |\mathbb{Q}|$.