Summer Session Week 7

Notes

August 2021

- Intuitively, the *cardinality* of a set is how "big" it is. If A is a set, we let $|A|$ denote its cardinality, read "the cardinality of A".

- First, we will talk about finite sets. How many elements is in the set $A = \{3, \pi, \text{dog}, \square\}$? The answer is four. But how did we get there? Simple: we counted the elements. Now, let us dive a bit deeper into what it means to count.

- By counting the set above, we mentally constructed a bijection between the set $B = \{1, 2, 3, 4\}$ and A. That bijection can be defined explicitly as $f : B \to A, f(1) = 3, f(2) = \pi, f(3) = \text{dog}, f(4) = \square$. This is the essence of cardinality.

- Let $\mathbb{N}_n$ be the first n counting numbers, i.e. $\mathbb{N}_n = \{1, 2, \ldots, n\}$. We will define $|\mathbb{N}_n| = n$. We will also say a set A has cardinality n, or $|A| = n$, if there exists a bijection between $\mathbb{N}_n$ and A. Fact: if there is a bijection between A and $\mathbb{N}_n$ for some n, then there is no bijection between A and $\mathbb{N}_m$ for any $m \neq n$. That is, the cardinality of a finite set is unique.

- Now, let us generalize to infinite sets. A set is called infinite if it is not finite. That is, there is no bijection to $\mathbb{N}_n$ for any n. There are two types of infinities we will care about here: countable infinity and uncountable infinity.

- A set is *countable* if there exists a bijection between the set and $\mathbb{N}$. The name countable comes from the fact that the set $\mathbb{N}$ is the set of counting numbers. That is, if a set A is countable, we can "count" the elements of A. This is, by construction of $\mathbb{N}$, the smallest infinity. We define $|\mathbb{N}| = \aleph_0$. This is the first letter of the Hebrew alphabet, read "aleph".

- If $|A| = n$, then we can write $A = \{a_1, a_2, \ldots, a_n\}$. If $|A| = \aleph_0$, then we can write $A = \{a_1, a_2, \ldots\}$. The map $f : \mathbb{N}_n \to A$ or $f : \mathbb{N} \to A$, where $f(n) = a_n$, will then be a bijection. If the cardinality of A is bigger than $\aleph_0$ (these sets exists, and we will prove it next), then it is impossible to write A as above.

- If a set is infinite and not countable, it is said to be *uncountable*. The classic example of this is $\mathbb{R}$. We will prove this using Cantor's diagonalization argument. First, a quick fact: $|\mathbb{R}| = |(0, 1)|$ (see exercises). For the sake of contradiction, assume that we could count the elements of $(0, 1)$. We will write each element of $(0, 1)$ using its decimal form. By assumption, we can do the following: let a_n be the nth element of $(0, 1)$. Then, write
  $a_1 = 0.a_1^1 a_1^2 a_1^3 \ldots$
  $a_2 = 0.a_2^1 a_2^2 a_2^3 \ldots$
  $a_3 = 0.a_3^1 a_3^2 a_3^3 \ldots$
  $\vdots$
  Now let b be the number $0.b_1 b_2 b_3 \ldots$, where $b_i = a_i^i + 1$ if $a_i^i \leq 8$, or $b_i = a_i^i - 1$ if $a_i^i = 9$. This is clearly a real number in $(0, 1)$, so it should be somewhere in our list. It cannot be a_1, as they have a different first digit. It cannot be a_2, as they have a different second digit. In fact, it cannot be any a_n, as they have a different nth digit. So we have a contradiction. Thus we cannot count the real numbers.

- The symbol for the cardinality of the real numbers is $|\mathbb{R}| = \mathfrak{c}$. It is a non-trivial fact that $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$. You may ask "are there any cardinal numbers between $\aleph_0$ and $\mathfrak{c}$?" This is called the *continuum hypothesis*, and it is well known that it is impossible to prove that this is true or that it is false. This is very difficult, and requires a lot of axiomatic set theory and logic.

- Now, how do we compare the sizes of sets? We use injective and surjective functions. Intuitively, if we can find and injection from A into B, then the set B must have at least as many elements as A. If we can find a surjection from A onto B, then A should have at least as many elements as B. We formalize this as a definition.

- We write $|A| \leq |B|$ if there exists an injection from A into B. We write $|A| \geq |B|$ if there exists a surjection from A onto B. Finally, we write $|A| = |B|$ if there exists a bijection from A onto B. We can also write $|A| < |B|$ if there is an injection from A into B, but there does not exist a bijection between them. Similarly for $>$.

- In the exercises, you will prove that the symbols $\leq, <, =, >, \geq$ work exactly as you would think. In fact, we have already proven most of it in previous weeks.

- Now, let us move on to cardinal arithmetic. Assume A, B are finite sets. **Theorem**: If $A \cap B = \emptyset$, or A and B are disjoint, then $|A \cup B| = |A| + |B|$. See end of notes for a proof. Also, we have proved earlier that $|A \times B| = |A| \cdot |B|$. For infinite sets, we take this as the definition.

- We define $|A| + |B| = |A \cup B|$ if A and B are disjoint. If they are not we can make them disjoint by replacing A with $A \times \{0\}$ and B with $B \times \{1\}$. We define $|A| \cdot |B| = |A \times B|$.

- We will now prove **Theorem**. First, we will need a lemma: If sets A, B, C, D are finite, $A \cap B = C \cap D = \emptyset$, and there are bijections $f_1 : A \to C$ and $f_2 : B \to D$, then there is a bijection from $A \cup B$ to $C \cup D$. **Proof**: define the map $g : A \cup B \to C \cup D$ by

$$g(x) = \begin{cases} f_1(x) & \text{if } x \in A \\ f_2(x) & \text{if } x \in B \end{cases}$$

First we show that g is injective. Let $x_1, x_2 \in A \cup B$, and $x_1 \neq x_2$. If $x_1, x_2 \in A$, then $g(x_1) = f_1(x_1) \neq f_1(x_2) = g(x_2)$ by the injectivity of f_1. The case where $x_1, x_2 \in B$ is similar. So the last case is $x_1 \in A$ and $x_2 \in B$. In this case, $g(x_1) = f_1(x_1) \in C$ and $g(x_2) = f_2(x_2) \in D$. As C and D are disjoint, $g(x_1) \neq g(x_2)$. Now, we will show that g is surjective. Let $y \in C \cup D$. If $y \in C$, by surjectivity of f_1, there is an $x \in A \subset A \cup B$ such that $f_1(x) = y$. Similar if $y \in D$. Thus g is surjective, and hence bijective.
Now we will prove the theorem. The result is trivial if either set is empty, so suppose neither are. Let $|A| = n$ and $|B| = m$, and let f be the bijection between $\mathbb{N}_n$ and A, and g be the bijection between $\mathbb{N}_m$ and B. Let $h : \mathbb{N}_n \to \mathbb{N}_{n+m} \setminus \mathbb{N}_m$, defined by $h(i) = m + i$ for $i \in \mathbb{N}_n$. It is an exercise to the reader to show that h is bijective. Thus the function $f \circ h^{-1}$ is a bijection between $\mathbb{N}_{n+m} \setminus \mathbb{N}_m$ to A. Now, define another function $\phi : \mathbb{N}_{n+m} \setminus \mathbb{N}_m \cup \mathbb{N}_m \to A \cup B$, with

$$\phi(x) = \begin{cases} (f \circ h^{-1})(x) & \text{if } x \in \mathbb{N}_{n+m} \setminus \mathbb{N}_m \\ g(i) & \text{if } i \in \mathbb{N}_m \end{cases}$$

By the previous lemma, ϕ is a bijection. Its domain is simply $\mathbb{N}_{n+m}$, and codomain is $A \cup B$. Thus $|A \cup B| = |\mathbb{N}_{n+m}| = n + m = |A| + |B|$.

- Lastly, if we have a family of sets U_i, where $i \in \mathbb{N}_n$ for some n, call this family *mutually (pairwise) disjoint* if, whenever $i \neq j, U_i \cap U_j = \emptyset$. In this case, if we take the union over all of the sets, written $\bigcup_{i=1}^{n} U_i$, we have $|\bigcup_{i=1}^{n} U_i| = \sum_{i=1}^{n} |U_i|$, where the $\sum$ symbol means the sum of all the items. This can be proved using induction.

Summer Session Week 7

Exercises

August 2021

You are allowed to use any exercises from previous weeks, even ones you did not solve.

Exercise 1: Define $|A| - |B| = |C|$ if and only if $|A| = |C| + |B|$. Suppose $B \subset A$. Prove that $|A \setminus B| = |A| - |B|$. Why do we need $B \subset A$? (Hint: draw the Venn diagram and use theorem.)
Exercise 2: Prove $|A \cup B| = |A| + |B| - |A \cap B|$. Why does this not contradict theorem? (Hint: look at the hint for the previous problem, and use the previous problem.)
***Exercise 3**: Prove $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$. This and the previous problem are special cases of what is called the "principle of inclusion-exclusion". Can you spot a pattern? Take a guess for what the formula will look like for four sets. (Hint: same as before.)
Exercise 4: Prove $|A| \leq |\mathcal{P}(A)|$.
***Exercise 5**: Prove $|A| < |\mathcal{P}(A)|$. (Hint: Make the set $R = \{a \in A | a \notin f(a)\}$, and show f cannot be surjective. I hope you remember what we did with Russel's paradox!)
Exercise 6: Prove that if $|A| \leq |B|$, then $|B| \geq |A|$.
Exercise 7: Prove that if $|A| \geq |B|$, then $|B| \leq |A|$.

A relation $\leq$ is called a "strong partial order" if the following axioms are satisfied: For any a, b, c, we have
1. $a \leq a$ (reflexivity)
2. If $a \leq b$ and $b \leq a$, then $a = b$ (antisymmetry)
3. If $a \leq b$ and $b \leq c$, then $a \leq c$ (transitivity)
If, in addition, we assume that either $a \leq b$ or $b \leq a$ (strongly connected), then the relation is called a "total order"

Exercise 8: Prove that $\leq$ for cardinal numbers is reflexive.

Exercise 9: Prove that $\leq$ for cardinal numbers is antisymmetric.
Exercise 10: Prove that $\leq$ for cardinal numbers is transitive.

It is true that $\leq$ is strongly connected, but that is a LOT more complicated and outside our scope.

A relation $<$ is called a "weak partial order" if the following axioms are satisfied: For any a, b, c, we have
1. It is not true that $a < a$ (irreflexivity)
2. If $a < b$, then it is not the case that $b < a$ (asymmetry)
3. If $a < b$ and $b < c$, then $a < c$ (transitivity)

Exercise 11: Prove that $<$ for cardinal numbers is irreflexive.
Exercise 12: Prove that $<$ for cardinal numbers is asymmetric.
Exercise 13: Prove that $<$ for cardinal numbers is transitive.

A relation $\sim$ is called an "equivalence relation" if the following axioms are satisfied: For any a, b, c, we have
1. $a \sim a$ (reflexivity)
2. $a \sim b$ if and only if $b \sim a$ (symmetry)
3. If $a \sim b$ and $b \sim c$, then $a \sim c$ (transitivity)
An equivalence relation is the closest we can get to two things being "equal" without actually being equal.

Exercise 14: Prove that $=$ for cardinal numbers is reflexive.
Exercise 15: Prove that $=$ for cardinal numbers is symmetric.
Exercise 16: Prove that $=$ for cardinal numbers is transitive.

Exercise 17: Prove that if $|A| = |B|$ and $|C| = |D|$, then $|A \times C| = |B \times D|$. (Hint: define a bijection that takes in an ordered pair, and outputs an ordered pair, but with different bijections applied to each coordinate.)
Exercise 18: Prove $|\mathbb{N}| = |\mathbb{Z}|$.
Exercise 19: Prove $|(0, 1) = |(a, b)|$, for any $a < b$.
Exercise 20: Prove that, if A is an infinite set, then $|A| = |A \cup \{a\}|$ for any object a. You may assume that any infinite set has a countable subset, or you can try to prove this fact yourself (Hint: induction).
Exercise 21: Prove $|(0, 1)| = |(0, 1]| = |[0, 1)|$.
Exercise 22: If you know trigonometry, use the tangent or cotangent function (both are bijective if you restrict the domain correctly) to prove that $|\mathbb{R}| = |(0, 1)|$.
Exercise 23: Prove $|\mathbb{R}| = |(0, \infty)|$.

***Exercise 24**: Prove $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$. (Hint: Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, f(m,n) = 2^m \cdot 3^n$ to get an injection one direction. Then find an injection the other direction, and use exercise 9.)
***Exercise 25**: Prove $|\mathbb{N}| = |\mathbb{Q}|$. (Hint: build a surjection from $\mathbb{N}$ onto $\mathbb{Q}$ by making an infinite grid with an origin, and labeling the vertices $\frac{a}{b}$ if the vertex has the coordinate (a, b), unless $b = 0$, where you just leave it blank. Now, what f does is it makes a spiral, starting at the origin, going right one, up one, left two, down two, right three, etc. Draw a picture to help visualize this. You need not explicitly write out what f is.)