Summer Session Week 6

Solutions

August 2021

Exercise 1: $(f \circ g)(x) = 6x + 8 \qquad (g \circ f)(x) = 6x + 4$
Exercise 2: $(f \circ g)(x) = x - 6\sqrt{x} + 11 \qquad (g \circ f)(x) = \sqrt{2 + x^2} - 3$
Exercise 3: $(f \circ g)(x) = 5 - \frac{2}{\sqrt{x}}$, so $((f \circ g) \circ h)(x) = 5 - \frac{2}{x^{3/2}}$
$(g \circ h)(x) = \frac{2}{x^{3/2}}$, so $(f \circ (g \circ h))(x) = 5 - \frac{2}{x^{3/2}}$
Exercise 4: Let $f(x) = 2x$ and $g(x) = x + 4$, then $h(x) = g(f(x))$.
Exercise 5: Let $f(x) = x+1$ and $g(x) = x^2$, then $h(x) = g(f(x)) = (x+1)^2$.
Exercise 6: Let $f(x) = 3x$, $g(x) = \sqrt{x}$, and $l(x) = \frac{2}{x}$. then $h(x) = l(g(f(x)))$.
Exercise 7: We will write $x = \frac{2y+3}{y-4}$ and solve for y. First, multiply both sides by $y - 4$, then isolate the y, and divide. We get $x(y-4) = 2y+3$, then $y(x-2) = 3 + 4x$, and lastly $y = \frac{3+4x}{x-2}$. So let $f^{-1}(x) = \frac{3+4x}{x-2}$. Then we calculate: $(f \circ f^{-1})(x) = \frac{2\frac{3+4x}{x-2}+3}{\frac{3+4x}{x-2}-4} = \frac{6+8x+3x-6}{3+4x-4x+8} = \frac{11x}{11} = x$. Finding $(f^{-1} \circ f)(x)$ is similar.
Exercise 8: No, as f is not bijective. For example, $f(-1) = f(1) = 1$, but $1 \neq -1$.
Exercise 9: $f^{-1}(\{0,1,2\}) = \{0, 1, -1, \sqrt{2}, -\sqrt{2}\}$, and $f^{-1}(\{-1\}) = \emptyset$.
Exercise 10: Make both the domain and codomain the set of real numbers greater than or equal to 0; $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. Now $f(x) = x^2$ is bijective, and $f^{-1}(x) = \sqrt{x}$.
Exercise 11: Suppose $A \subset B$, and let $x \in f^{-1}(A)$. That means $f(x) \in A$, and so $f(x) \in B$ too. But that means $x \in f^{-1}(B)$, so $f^{-1}(A) \subset f^{-1}(B)$.
Exercise 12: Let $x \in f^{-1}(A \cup B)$. That means $f(x) \in A \cup B$. There are two cases. First suppose $f(x) \in A$. Then $x \in f^{-1}(A)$, and so $x \in f^{-1}(A) \cup f^{-1}(B)$. If instead $f(x) \in B$, then we have $x \in f^{-1}(B)$, and again we get $x \in f^{-1}(A) \cup f^{-1}(B)$. Thus $f^{-1}(A \cup B) \subset f^{-1}(A) \cup f^{-1}(B)$.
Now suppose $x \in f^{-1}(A) \cup f^{-1}(B)$. If $x \in f^{-1}(A)$, then $f(x) \in A$, so $f(x) \in A \cup B$, and $x \in f^{-1}(A \cup B)$. If instead $x \in f^{-1}(B)$, then $f(x) \in B$, so $f(x) \in A \cup B$, and again $x \in f^{-1}(A \cup B)$. Thus $f^{-1}(A) \cup f^{-1}(B) \subset f^{-1}(A \cup B)$,

and hence equality.
Exercise 13: Let $x \in f^{-1}(A \cap B)$. Then $f(x) \in A \cap B$, so $f(x) \in A$, which implies $x \in f^{-1}(A)$, and $f(x) \in B$, so $x \in f^{-1}(B)$. Thus $x \in f^{-1}(A) \cap f^{-1}(B)$. So $f^{-1}(A \cap B) \subset f^{-1}(A) \cap f^{-1}(B)$.
Now let $x \in f^{-1}(A) \cap f^{-1}(B)$. Then, as $x \in f^{-1}(A)$, $f(x) \in A$, and as $x \in f^{-1}(B), f(x) \in B$. Thus $f(x) \in A \cap B$, so $x \in f^{-1}(A \cap B)$. Thus $f^{-1}(A) \cap f^{-1}(B) \subset f^{-1}(A \cap B)$, and we have equality.
Exercise 14: Let $x \in f^{-1}(A \setminus B)$. Then $f(x) \in A \setminus B$, so $f(x) \in A$ and $f(x) \notin B$. From the first one, $x \in f^{-1}(A)$. Suppose $x \in f^{-1}(B)$, then $f(x) \in B$, contradiction. Thus $x \notin f^{-1}(B)$, so $x \in f^{-1}(A) \setminus f^{-1}(B)$, and $f^{-1}(A \setminus B) \subset f^{-1}(A) \setminus f^{-1}(B)$.
Now let $x \in f^{-1}(A) \setminus f^{-1}(B)$, so that $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$. That means $f(x) \in A$. Suppose $f(x) \in B$, then $x \in f^{-1}(B)$, contradiction. So $f(x) \notin B$, and $f(x) \in A \setminus B$, so $x \in f^{-1}(A \setminus B)$. Thus $f^{-1}(A) \setminus f^{-1}(B) \subset f^{-1}(A \setminus B)$, and so equality.
Exercise 15: $f^{-1}(A \triangle B) = f^{-1}((A \cup B) \setminus (A \cap B)) = f^{-1}(A \cup B) \setminus f^{-1}(A \cap B) = (f^{-1}(A) \cup f^{-1}(B)) \setminus (f^{-1}(A) \cap f^{-1}(B)) = f^{-1}(A) \triangle f^{-1}(B)$.
Exercise 16: Suppose $(f \circ g)(x) = (f \circ g)(y)$, then $f(g(x)) = f(g(y))$, and by injectivity of f, $g(x) = g(y)$, and by injectivity of g, $x = y$.
Exercise 17: Let $z \in Z$ (the codomain of the composition). As f is surjective, there is a $y \in Y$ such that $f(y) = z$. By surjectivity of g, we get that there is an $x \in X$ such that $g(x) = y$. This x works, as $(f \circ g)(x) = z$.
Exercise 18: Yes, as a function is bijective if an only if it is both injective and surjective.
Exercise 19: Suppose $g(x) = g(y)$. Then, we have $f(g(x)) = f(g(y))$, so $(f \circ g)(x) = (f \circ g)(y)$, and by injectivity of $f \circ g$, $x = y$.
***Exercise 20**: Let $g : \mathbb{R} \to \mathbb{R} \times \mathbb{R}, g(x) = (x, 0)$, and let $f : \mathbb{R} \times \mathbb{R}, f(x, y) = x$. Then $f \circ g : \mathbb{R} \to \mathbb{R}, (f \circ g)(x) = x$, so $f \circ g$ is bijective. Clearly g is not surjective as, for example, the point $(1, 1)$ is not in the image.
Exercise 21: Let $z \in Z$. By surjectivity of $f \circ g$, we can find an x such that $(f \circ g)(x) = z$. That is, $f(g(x)) = z$, so $g(x)$ works as our input to f to get z.
***Exercise 22**: The same functions from exercise 20 works, as $f(1, 0) = f(1, 1)$, but $(1, 0) \neq (1, 1)$.
***Exercise 23**: Suppose there are two functions that act as inverses, call them g and h. That is, $(f \circ g)(x) = (g \circ f)(x) = x$, and similar for h. We show $g = h$ by showing $g(x) = h(x)$ for all x in the domain. For any x, we have $g(x) = (h \circ f)(g(x)) = ((h \circ f) \circ g)(x) = (h \circ (f \circ g))(x) = h((f \circ g)(x)) = h(x)$.
***Exercise 24**: Suppose that $f : X \to Y$ is a function, and there exists an inverse, $f^{-1} : Y \to X$. Let $y \in Y$. Then $f^{-1}(y) = x$ for some x, so $f(x) = y$, and f is surjective. Suppose $f(x) = f(y)$, then $f^{-1}(f(x)) = f^{-1}(f(y))$, so

$x = y$, and f is injective. Thus f is bijective. f^{-1} is also bijective.
Now suppose $f : X \to Y$ is bijective. We will define a new function $g : Y \to X$, where $g(y) = x$ if and only if $f(x) = y$. As f is surjective, g is defined everywhere on its domain (and is surjective too). Now we show that one input of g only gives one output. Assume $g(y) = x$ and $g(y) = z$. That means $f(x) = y = f(z)$, but by the injectivity of f, $x - z$. Thus g is a function defined everywhere on its domain, and gives only one output per input, so it is indeed a function. Now, $f(g(y)) = f(x) = y$ and $g(f(x)) = g(y) = x$, so g is an inverse function to f. As inverse functions are unique, $g = f^{-1}$.