Summer Session Week 6

Notes

July 2021

- Recall that a function takes points in one set and moves them to points in another set. Say we are interested in moving points from a set X to a set Y using the rule $f : X \to Y$, then from Y to a set Z using the rule $g : Y \to Z$. There are two ways to go about this. Either we can iterate, first applying f to our point x, then applying g to the image, or we can find a new function $h : X \to Z$ that moves x directly to its image in Z, bypassing the set Y. This function h can be constructed using both f and g, and is called the *composition* of f and g.

- Let $f : X \to Y^*$, and $g : Y \to Z$, where $f(X) \subset Y$. Then we can form the composition function, $h : X \to Z$, $h(x) = (g \circ f)(x) = g(f(x))$. $g \circ f$ is read "g composed with f". Note that we need the condition $f(X) \subset Y$ so that h is defined everywhere on X.

- Example: Let $f, g : \mathbb{R} \to \mathbb{R}, f(x) = 2x, g(x) = x + 3$. Then $(f \circ g)(x) = f(g(x)) = f(x + 3) = 2(x + 3) = 2x + 6$. We also have $(g \circ f)(x) = g(f(x)) = g(2x) = (2x) + 3 = 2x + 3$. Notice that $f \circ g \neq g \circ f$. This is true in general.

- Example: Let $h : \mathbb{R} \setminus \{0\} \to \mathbb{R}, h(x) = \frac{1}{x}$. Using the f and g from before, we have $(g \circ h)(x) = g(h(x)) = g(\frac{1}{x}) = \frac{1}{x} + 3$. Therefore, $(f \circ (g \circ h))(x) = f(\frac{1}{x} + 3) = 2(\frac{1}{x} + 3) = \frac{2}{x} + 6$. We have previously found $(f \circ g)(x) = 2x + 6$, so $((f \circ g) \circ h)(x) = (f \circ g)(h(x)) = (f \circ g)(\frac{1}{x}) = \frac{2}{x} + 6$. So here, $(f \circ g) \circ h = f \circ (g \circ h)$.

- Composition of functions is always associative, as we found in the previous example. Here is a quick proof: $(f \circ (g \circ h))(x) = f \circ (g \circ h)(x) = f \circ g(h(x)) = f(g(h(x))) = (f \circ g)(h(x)) = ((f \circ g) \circ h)(x)$.

- Let $f : X \to Y, A \subset X$. Recall the definition for the image of a set: $f(A) = \{f(x) | x \in A\}$. Let us now go backwards.

- Let $B \subset Y$. Then the *inverse image* of B under f is $f^{-1}(B) = \{x \in X | f(x) \in B\}$. This is the set of all points that, after moving with f, live in B.

- If B is a singleton set, $B = \{y\}$, then the set $f^{-1}(B) = f^{-1}(\{y\})$ is called the *fiber* over y.

- The notation can be a little confusing, as f^{-1} also denotes the inverse function. If $f : X \to Y$, then $f^{-1} : Y \to X$, such that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$.

- If a function has an inverse, then $f(x) = y$ if and only if $f^{-1}(y) = x$.

- The inverse image of a set always exists, while most functions do not have an inverse. In fact, a function has an inverse if and only if it is bijective (exercise).

- Example: Let $f : \mathbb{R} \to \mathbb{R}, f(x) = 2x + 3$. Let $B = [3, 5]$. Then $f^{-1}(B) = [0, 1]$

- Example: In the previous example, f is bijective, so let us find its inverse. Write $y = f(x) = 2x + 3$, and we will solve for x. Subtract 3 from both sides then divide by 2 to get $x = \frac{1}{2}y - \frac{3}{2}$. Thus $f^{-1}(y) = \frac{1}{2}y - \frac{3}{2}$. Here, y is just a dummy variable, so we could write $f^{-1}(x)$ instead.

- Example: In the example above, we will show $f(f^{-1}(x)) = f^{-1}(f(x)) = x$. $f(f^{-1}(x)) = f(\frac{1}{2}x - \frac{3}{2}) = 2(\frac{1}{2}x - \frac{3}{2}) + 3 = x - 3 + 3 = x$. $f^{-1}(f(x)) = f^{-1}(2x + 3) = \frac{1}{2}(2x + 3) - \frac{3}{2} = x + \frac{3}{2} - \frac{3}{2} = x$.

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Exercises

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Do not worry about the domain and codomain for the following questions.

Exercise 1: Let $f(x) = 2x$ and $g(x) = 3x + 4$. Find $f \circ g$ and $g \circ f$.
Exercise 2: Let $f(x) = 2 + x^2$ and $g(x) = \sqrt{x} - 3$. Find $f \circ g$ and $g \circ f$.
Exercise 3: Let $f(x) = 5 - x$, $g(x) = \frac{2}{\sqrt{x}}$, and $h(x) = x^3$. Show that function composition is associative by finding $(f \circ g) \circ h$ and $f \circ (g \circ h)$.
Exercise 4: Write the following function as the composition of two functions: $h(x) = 2x + 4$.
Exercise 5: Write the following function as the composition of two functions: $h(x) = x^2 + 2x + 1$.
Exercise 6: Write the following function as the composition of three functions: $h(x) = \frac{2}{\sqrt{3x}}$.
Exercise 7: Let $f(x) = \frac{2x+3}{x-4}$. Take for granted that f is bijective. Find $f^{-1}(x)$. Show that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$

Let $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$.

Exercise 8: Does f have an inverse function? Why or why not.
Exercise 9: What is $f^{-1}(\{0, 1, 2\})$? What is $f^{-1}(\{-1\})$?
Exercise 10: Change the domain and codomain of f so that it is bijective. What is the inverse function in this case?

Let $f : X \to Y$, $A, B \subset Y$.

Exercise 11: Prove that if $A \subset B$, then $f^{-1}(A) \subset f^{-1}(B)$.
Exercise 12: Prove that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.
Exercise 13: Prove that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.
Exercise 14: Prove that $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$.

Exercise 15: Prove that $f^{-1}(A \triangle B) = f^{-1}(A) \triangle f^{-1}(B)$.

Note that these exercises show that the inverse image is a really nice function. For the forward image, we needed to assume injectivity for equality to hold. No such assumptions here.

Let $f : Y \to Z$ and $g : X \to Y$.

Exercise 16: Assume f and g are injective. Prove that $f \circ g$ is injective.
Exercise 17: Assume f and g are surjective. Prove that $f \circ g$ is surjective.
Exercise 18: If f and g are bijective, is $f \circ g$ necessarily bijective?
Exercise 19: Suppose $f \circ g$ is injective. Prove that g is also injective.
***Exercise 20**: Find functions f, g such that $f \circ g$ is bijective but g is not surjective.
Exercise 21: Suppose $f \circ g$ is surjective. Prove that f is also surjective.
***Exercise 22**: Find functions f, g such that $f \circ g$ is bijective but f is not injective.
***Exercise 23**: Prove that if a function has an inverse, the inverse is unique.
***Exercise 24**: Prove that the inverse of a function exists if and only if the function is bijective.