Summer Session Week 5

Solutions

July 2021

Exercise 1: Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. So, in particular, $x \in B$. So $A \cap B \subset B$.
Exercise 2: Suppose $B \subset A$. Let $x \in B$. Then $x \in A$, so $x \in A \cap B$. So $B \subset A \cap B$. By previous exercise, we then have $A \cap B = B$.
Now suppose $A \cap B = B$. Let $x \in B$. Then $x \in A \cap B (= B)$, so $x \in A$. So $B \subset A$.
Exercise 3: Let $x \in A$. Then $x \in A$ or $x \in B$, so $x \in A \cup B$. So $A \subset A \cup B$.
Exercise 4: Suppose $B \subset A$. Let $x \in A \cup B$. Then $x \in A$ or $x \in B$. If $x \in B$, then $x \in A$. The other case also has $x \in A$. So in both cases, $x \in A$, so $A \cup B \subset A$. By previous exercise, $A \cup B = A$.
Now suppose $A \cup B = A$. Let $x \in B$. Then $x \in A \cup B = A$, so $x \in A$, and we have $B \subset A$.
Exercise 5: Let $x \in (A \cup B)^\mathsf{c}$. Then $x \notin A \cup B$. Suppose, for contradiction, that $x \in A$. Then $x \in A \cup B$, a contradiction. So $x \notin A$. Similarly, $x \notin B$. So $x \in A^\mathsf{c}$ and $x \in B^\mathsf{c}$, so $x \in A^\mathsf{c} \cap B^\mathsf{c}$. So $(A \cup B)^\mathsf{c} \subset A^\mathsf{c} \cap B^\mathsf{c}$.
Now let $x \in A^\mathsf{c} \cap B^\mathsf{c}$. Then $x \in A^\mathsf{c}$ and $x \in B^\mathsf{c}$. So $x \notin A$ and $x \notin B$. Suppose, for contradiction, that $x \in A \cup B$. Then $x \in A$ or $x \in B$. If $x \in A$, then $x \in A$ and $x \notin A$, a contradiction. If $x \in B$, then $x \in B$ and $x \notin B$, a contradiction. So, in any case, $x \in A \cup B$ leads to a contradiction. Therefore $x \notin A \cup B$, or $x \in (A \cup B)^\mathsf{c}$. So $A^\mathsf{c} \cap B^\mathsf{c} \subset (A \cup B)^\mathsf{c}$. Hence we have equality.
Exercise 6: Let $x \in (A \cap B)^\mathsf{c}$. Then $x \notin A \cap B$. We will show that we must have $x \notin A$ or $x \notin B$, or both. If $x \notin A$, then we are good. If not, that is, $x \in A$, then we must have $x \notin B$. For if $x \in B$, then $x \in A \cap B$, a contradiction. Thus $x \notin A$ or $x \notin B$. In either case, $x \in A^\mathsf{c} \cup B^\mathsf{c}$, so $(A \cap B)^\mathsf{c} \subset A^\mathsf{c} \cup B^\mathsf{c}$.
Now let $x \in A^\mathsf{c} \cup B^\mathsf{c}$. If $x \in A^\mathsf{c}$, then $x \notin A \supset A \cap B$, so $x \notin A \cap B$ (for if $x \in A \cap B$, then $x \in A$, contradiction), and $x \in (A \cap B)^\mathsf{c}$. Similar if $x \in B^\mathsf{c}$. So $A^\mathsf{c} \cup B^\mathsf{c} \subset (A \cap B)^\mathsf{c}$. Hence we have equality.
Exercise 7: Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. If $x \in B$,

then $x \in A \cap B$, so $x \in (A \cap B) \cup (A \cap C)$. If $x \in C$, then $x \in A \cap B$, so $x \in (A \cap B) \cup (A \cap C)$. Hence $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$.
Now suppose $x \in (A \cap B) \cup (A \cap C)$. If $x \in A \cap B$, then $x \in A$, and $x \in B \subset B \cup C$, so $x \in A \cap (B \cup C)$. If $x \in A \cap C$, then $x \in A$ and $x \in C \subset B \cup C$, so $x \in A \cap (B \cup C)$. So $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$. Hence equality.
Exercise 8: Let $x \in A \cup (B \cap C)$. If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$, so $x \in (A \cup B) \cap (A \cup C)$. If $x \in B \cap C$, then $x \in B \subset A \cup B$ and $x \in C \subset A \cup C$, so $x \in (A \cup B) \cap (A \cup C)$. Hence $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$.
Now suppose $x \in (A \cup B) \cap (A \cup C)$. Then $x \in A \cup B$ and $x \in A \cup C$. In the first one, suppose $x \in A$. Then $x \in A \cup (B \cap C)$. The other case is $x \in B$. So suppose $x \in B$. Then from the second, either $x \in A$ or $x \in C$. If $x \in A$, then $x \in A \cup (B \cap C)$. If not, then $x \in C$. So $x \in B \cap C$, so $x \in A \cup (B \cap C)$. Therefore $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$. Hence equality.
Exercise 9: Suppose $A \subset B$. Let $x \in B^{\mathsf{c}}$. Suppose $x \in A$. Then $x \in B$, but that contradicts the fact that $x \in B^{\mathsf{c}}$. So $x \notin A$, so $x \in A^{\mathsf{c}}$, and $B^{\mathsf{c}} \subset A^{\mathsf{c}}$.
Now suppose $B^{\mathsf{c}} \subset A^{\mathsf{c}}$. Let $x \in A$. Suppose $x \notin B$, so that $x \in B^{\mathsf{c}}$. Then $x \in A^{\mathsf{c}}$, so $x \notin A$, which is a contradiction. So $x \in B$, and $A \subset B$.
Exercise 10: Let $(a, b) \in A \times (B \cap C)$. Then $a \in A$ and $b \in B \cap C$, so $b \in B$ and $b \in C$. Thus $(a, b) \in A \times B$ and $(a, b) \in A \times C$, so $(a, b) \in (A \times B) \cap (A \times C)$. So $A \times (B \cap C) \subset (A \times B) \cap (A \times C)$.
Now let $(a, b) \in (A \times B) \cap (A \times C)$, so that $(a, b) \in A \times B$ and $(a, b) \in A \times C$. From the first one, $a \in A$ and $b \in B$, and the second one gives us $b \in C$. Thus $b \in B \cap C$, so $(a, b) \in A \times (B \cap C)$. Thus $(A \times B) \cap (A \times C) \subset A \times (B \cap C)$. Hence equality.
Exercise 11: Let $(a, b) \in A \times (B \cup C)$. Then $a \in A$ and $b \in B \cup C$, so $b \in B$ or $b \in C$. If $b \in B$, then $(a, b) \in A \times B$, so $(a, b) \in (A \times B) \cup (A \times C)$. If $b \in C$, then $(a, b) \in A \times C$, so $(a, b) \in (A \times B) \cup (A \times C)$. As these are all the cases, we have $A \times (B \cup C) \subset (A \times B) \cup (A \times C)$.
Now suppose $(a, b) \in (A \times B) \cup (A \times C)$. Then $(a, b) \in A \times B$ or $(a, b) \in A \times C$. In the first case, $a \in A$ and $b \in B$, so $b \in B \cup C$. In the second case, $a \in A$ and $b \in C$, so $b \in B \cup C$. In either case, $a \in A$ and $b \in B \cup C$, so $(a, b) \in A \times (B \cup C)$. Thus $(A \times B) \cup (A \times C) \subset A \times (B \cup C)$. Hence equality.
Exercise 12: Let $(a, b) \in A \times (B \setminus C)$. Then $a \in A$ and $b \in B \setminus C$, so $b \in B$ and $b \notin C$. As $b \in B$, $(a, b) \in A \times B$. As $b \notin C$, $(a, b) \notin A \times C$. Thus $(a, b) \in (A \times B) \setminus (A \times C)$, and so $A \times (B \setminus C) \subset (A \times B) \setminus (A \times C)$.
Now suppose $(a, b) \in (A \times B) \setminus (A \times C)$. Then $(a, b) \in A \times B$, so $a \in A$ and $b \in B$. Also, $(a, b) \notin A \times C$, so either $a \notin A$ or $b \notin C$. As $a \in A$, we must have $b \notin C$. Thus $b \in B \setminus C$, so $(a, b) \in A \times (B \setminus C)$. And so $(A \times B) \setminus (A \times C) \subset A \times (B \setminus C)$, thus we have equality.
Exercise 13: Let $(a, c) \in (A \cap B) \times (C \cap D)$. Then $a \in A \cap B$ and

$c \in C \cap D$. Thus $a \in A$ and $c \in C$, so $(a, b) \in A \times C$. Also, $a \in B$ and $c \in D$, so $(a, c) \in B \times D$. So $(a, c) \in (A \times C) \cap (B \times D)$. Thus $(A \cap B) \times (C \cap D) \subset (A \times C) \cap (B \times D)$.
Now let $(a, c) \in (A \times C) \cap (B \times D)$. Then $(a, c) \in A \times C$, so $a \in A$ and $c \in C$. Also, $(a, c) \in B \times D$, so $a \in B$ and $c \in D$. Thus $a \in A \cap B$ and $c \in C \cap D$, so $(a, b) \in (A \cap B) \times (C \cap D)$. Hence $(A \times C) \cap (B \times D) \subset (A \cap B) \times (C \cap D)$, and we have equality.
Exercise 14: Suppose $A \subset B$, and let $(a, c) \in A \times C$. Then $a \in A$, and so $a \in B$. As $c \in C$, we have $(a, c) \in B \times C$, and so $A \times C \subset B \times C$.
Exercise 15: First suppose $A \times B \subset C \times D$. Suppose $a \in A$ and $b \in B$. Then $(a, b) \in A \times B$, so $(a, b) \in C \times D$. Thus $a \in C$ and $b \in D$. Hence $A \subset C$ and $B \subset D$.
Now suppose $A \subset C$ and $B \subset D$. Let $(a, b) \in A \times B$. Then $a \in A$, so $a \in C$. Also, $b \in B$, so $b \in D$. Thus $(a, b) \in C \times D$, and so $A \times B \subset C \times D$.
Exercise 16: As $X, Y \in \mathcal{P}(A)$, we have $X \subset A$ and $Y \subset A$. If $x \in X \cup Y$, then $x \in X$ (so $x \in A$) or $x \in Y$ (so $x \in A$), so $x \in A$. Therefore $X \cup Y \in \mathcal{P}(A)$. We also have $X \cap Y \subset X \subset A$, so $X \cap Y \in \mathcal{P}(A)$. Lastly, if $x \in X \setminus Y$, then $x \in X$, so $x \in A$, and $X \setminus Y \subset A$. Therefore $X \setminus Y \in \mathcal{P}(A)$.
Exercise 17: Let $y \in f(A)$. Then there is an $x \in A$ such that $f(x) = y$. As $A \subset B$, $x \in B$, so $y \in f(B)$. Thus $f(A) \subset f(B)$.
Exercise 18: Let $y \in f(A \cup B)$. Then there exists an $x \in A \cup B$ such that $f(x) = y$. There are two cases. First, suppose $x \in A$. Then $y \in f(A)$, so $y \in f(A) \cup f(B)$. If instead $x \in B, y \in f(B)$, so $y \in f(A) \cup f(B)$. Thus $f(A \cup B) \subset f(A) \cup f(B)$.
Exercise 19: Let $y \in f(A) \setminus f(B)$. Then $y \in f(A)$, so there is an $x \in A$ such that $f(x) = y$. As $y \notin f(B)$, there is no $a \in B$ such that $f(a) = y$, so $x \notin B$. Thus $x \in A \setminus B$, and so $y \in f(A \setminus B)$.
Exercise 20: Let $f : \mathbb{N} \to \mathbb{N}$, where $f(x) = x^2$. Let $A = \{-1, 0, 1\}$ and $B = \{-1\}$. Then $f(A \setminus B) = f(\{0, 1\}) = \{0, 1\}$. We have $f(A) = \{0, 1\}$ and $f(B) = \{1\}$, so $f(A) \setminus f(B) = \{0\}$.
Exercise 21: We just need the other set inclusion. Suppose $y \in f(A \setminus B)$. Then there is an $x \in A \setminus B$ such that $f(x) = y$. As $x \in A$, $y \in f(A)$. To show $y \notin f(B)$, suppose there were a $z \in B$ such that $f(z) = y$. Then $f(x) = f(z)$, so $x = z$. But as $z \in B, x \in B$, contradiction. Thus there is no $z \in B$ such that $f(z) = y$, and we have $y \notin f(B)$. Thus $y \in f(A) \setminus f(B)$.
***Exercise 22**: Let c be any fixed element of X. Let $y \in Y$. Then either $y \in f(X)$ or $y \notin f(X)$. If $y \in f(X)$, then there is only one $x \in X$ such that $f(x) = y$. So define g such that $g(y) = x$. If $y \notin f(X)$, then let $g(y) = c$. Now we have g defined everywhere on Y, and every element of X is the image

of some element of Y by g. To recap, we have:

$$g(y) = \begin{cases} x \text{ so that } f(x) = y & \text{if } y \in f(X) \\ c & \text{if } y \notin f(X) \end{cases}$$

***Exercise 23**: Let $y \in Y$. Look at the set $f^{-1}(y) = \{x \in X | f(x) = y\}$. For each $y \in Y$, pick one element (call it x) out of the corresponding set (to do this, we technically need something called the Axiom of Choice, but that is beyond our scope here). Define g so that $g(y) = x$. Then g is injective. To recap, we have: $g(y) = x$, for any fixed x such that $f(x) = y$.