Summer Session Week 5

Notes

July 2021

- Suppose the letters "p" and "q" stand for mathematical or logical sentences. For example, we may have "p" stand for "$a^2 + b^2 = c^2$".
- When doing a proof of the form "if p, then q", assume p is true, do some math and logic, then show that q follows.
- Example: If $2x + 3 = 5$, then $x = 1$. We assume $2x + 3 = 5$. Then, by subtracting 3 from both sides, we get $2x = 2$. Then we divide both sides by 2 to get $x = 1$.
- When doing a proof of the form "p if and only if q", we need to do two proofs. We need to prove both "if p, then q" and "if q then p".
- Example: $2x + 3 = 5$ if and only if $x = 1$. We have already shown if $2x + 3 = 5$, then $x = 1$. Now assume $x = 1$. Multiply both sides by 2 to get $2x = 2$, then add 3 to both sides to get $2x + 3 = 5$.
- Just because we have proved "if p, then q", that does not mean it is possible to prove "if q, then p". For example, if a shape is a square, it is also a rectangle. But just because a shape is a rectangle, there is no reason it must also be a square.
- If you want to prove something is false, or that there does not exist some sort of object, a powerful tool is proof by contradiction. If we are proving "if p, then q" by contradiction, assume "p" and "not q" are true, and do math and logic to arrive at a contradiction. A contradiction is showing both "r" and "not r" are true at the same time. We conclude that "q" is then true.

- Example: If $2x + 3 = 5$, then $x \neq 2$. For the sake of contradiction, assume $x = 2$. Then by multiplying both sides by 2 and adding 3, we get $2x+3 = 7$. So we have both $2x+3 = 5$ and $2x+3 = 7$. But $5 \neq 7$, so $2x + 3 = 5$ and $2x + 3 \neq 5$, a contradiction. Thus $x \neq 2$.

- When doing any sort of proof, always write out your hypotheses first. If you are doing an "if and only if" proof, make it known to the reader which direction you are doing first. Do this by writing either "assume p is true" or "assume q is true".

- Sometimes when writing a proof, we will have to break up the problem into two or more cases. Always make it known to the reader that you are treating each case individually, and let it be known which case you are working on.

- Example: If $x^2 = 1$, then $x + 1 = 0$ or $x + 1 = 2$. Suppose $x^2 = 1$. Then $x = 1$ or $x = -1$. These are the two cases. First suppose that $x = 1$. Then $x + 1 = 2$. Now suppose $x = -1$. Then $x + 1 = 0$. So in either case, the conclusion holds.

- We will now go over some implications from different set theoretical assumptions.

- If we assume $x \in A \cap B$, then we have $x \in A$ and $x \in B$. Conversely, if $x \in A$ and $x \in B$, then we have $x \in A \cap B$ (we could have also said that $x \in A \cap B$ if and only if $x \in A$ and $x \in B$).

- If we assume $x \in A$, then we have $x \in A \cup B$ for any set B. Conversely, if $x \in A \cup B$, we have either $x \in A$ or $x \in B$. There are two cases. You will have to work on both cases separately.

- $A \cap B = B \cap A$ and $A \cup B = B \cup A$.

- If $x \notin A$, then $x \in A^{\mathsf{c}}$. If $x \in A^{\mathsf{c}}$, then $x \notin A$.

- If $x \in A \setminus B$, then $x \in A$ and $x \notin B$. Conversely, if $x \in A$ and $x \notin B$, we have $x \in A \setminus B$.

- If $(a, b) \in A \times B$, then $a \in A$ and $b \in B$. Conversely, if $a \in A$ and $b \in B$, then $(a, b) \in A \times B$.

- If $x \in \mathcal{P}(A)$, then $x \subset A$. Conversely, if $x \subset A$, then $x \in \mathcal{P}(A)$.

- If $x \in f(A)$, then there is some $a \in A$ such that $f(a) = x$. Conversely, if there is some $a \in A$ such that $f(a) = x$, then $x \in f(A)$.

- A function $f : X \to Y$ is surjective if and only if $f(X) = Y$.

- A function $f : X \to Y$ is injective if and only if, whenever $f(a) = f(b)$, it must be the case that $a = b$.

- function $f : X \to Y$ is bijective if and only if it is both injective and surjective.

- To prove $A \subset B$, assume $x \in A$. Then, after doing math and logic, show that $x \in B$. Conclude that $A \subset B$. Conversely, if $A \subset B$ and $x \in A$, we then have that $x \in B$.

- To prove $A = B$, prove both that $A \subset B$ and $B \subset A$.

- Example: Prove $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$. Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. As $x \in B \cup C$, we have either $x \in B$ or $x \in C$. First, suppose $x \in B$. Then we have both $x \in A$ and $x \in B$, so $x \in A \cap B$. Therefore, $x \in (A \cap B) \cup (A \cap C)$. For the other case, assume $x \in C$. Then $x \in A$ and $x \in C$, so $x \in A \cap C$. Thus $x \in (A \cap B) \cup (A \cap C)$. So, in any case, we have $x \in (A \cap B) \cup (A \cap C)$. Thus $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$.

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Exercises

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Exercise 1: Prove $A \cap B \subset B$.
Exercise 2: Prove $A \cap B = B$ if and only if $B \subset A$.
Exercise 3: Prove $A \cup B \supset A$.
Exercise 4: Prove $A \cup B = A$ if and only if $B \subset A$.
Exercise 5: Prove $(A \cup B)^{\mathsf{c}} = A^{\mathsf{c}} \cap B^{\mathsf{c}}$ (Hint: you will need contradiction).
Exercise 6: Prove $(A \cap B)^{\mathsf{c}} = A^{\mathsf{c}} \cup B^{\mathsf{c}}$ (Hint: you will need contradiction).
Exercise 7: Prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
Exercise 8: Prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
Exercise 9: Prove $A \subset B$ if and only if $B^{\mathsf{c}} \subset A^{\mathsf{c}}$ (Use contradiction).
Exercise 10: Prove $A \times (B \cap C) = (A \times B) \cap (A \times C)$.
Exercise 11: Prove $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
Exercise 12: Prove $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$.
Exercise 13: Prove $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$.
Exercise 14: Prove that if $A \subset B$, then $A \times C \subset B \times C$.
Exercise 15: Prove $A \times B \subset C \times D$ if and only if $A \subset C$ and $B \subset D$.
Exercise 16: If $X, Y \in \mathcal{P}(A)$, prove $X \cup Y \in \mathcal{P}(A)$, $X \cap Y \in \mathcal{P}(A)$, and $X \setminus Y \in \mathcal{P}(A)$
Exercise 17: Suppose $A \subset B \subset X$. Prove $f(A) \subset f(B)$.
Exercise 18: Suppose $A, B \subset X$. Prove $f(A \cup B) = f(A) \cup f(B)$.
Exercise 19: Suppose $A, B \subset X$. Prove that $f(A \setminus B) \supset f(A) \setminus f(B)$.
Exercise 20: In the context of the previous problem, show that it is possible to have $f(A \setminus B) \neq f(A) \setminus f(B)$.
Exercise 21: In the context of the previous two problems, suppose furthermore that f is injective. Prove then that $f(A \setminus B) = f(A) \setminus f(B)$.
***Exercise 22**: Suppose $f : X \to Y$ is injective. Prove that there exists a surjective function $g : Y \to X$.
***Exercise 23**: Suppose $f : X \to Y$ is surjective. Prove that there exists a injective function $g : Y \to X$.