Summer Session Week 4

Solutions

July 2021

Exercise 1: f_3 is not a well defined function, as $f_3(0) = 0$, but 0 is not in the codomain.
Exercise 2: f_2 and f_4 are injective. Suppose $f(a) = f(b)$ for $f = f_2$ or f_3. Then $a^2 = b^2$, so $a = \pm b$. But as $a, b \in \mathbb{N}, a, b > 0$, so $a = b$. None of the functions are surjective as there is no $x \in \mathbb{N}$ such that $x^2 = 2$ (we may prove this in a future week).
Exercise 3: $f_1(\{-3, -1, 1, 3\}) = \{1, 9\}$?
Exercise 4: First we will show f is injective. Suppose $f(a) = f(b)$. Then either both a and b are less than 1, or both greater than or equal to one. For if not, say $a < 1$ and $b \geq 1$, then we have $2a - 3 = 5b - 6$, or $a = \frac{5}{2}b - \frac{3}{2}$. But as $b \geq 1, a \geq \frac{5}{2} - \frac{3}{2} = 1$, contradiction. First suppose both $a, b < 1$, then $f(a) = f(b)$ implies $2a - 3 = 2b - 3$ which then implies $a = b$. Now suppose both $a, b \geq 1$, so we have $5a - 6 = 5b - 6$, so $a = b$. Thus f is injective.
Now we will show f is surjective. Let $y \in \mathbb{R}$. We have $y < -1$ or $y \geq -1$. If $y < -1$, we want an x so that $2x - 3 = y$. Letting $x = \frac{1}{2}(y + 3)$ works. Note as $y < -1, x < \frac{1}{2}(-1 + 3) = 1$, so we are in the first part of the piecewise definition. Now suppose $y \geq -1$. We can let $x = \frac{1}{5}(y + 6)$. In this case, $x \geq \frac{1}{5}(-1 + 6) = 1$, so we are in the second part of the piecewise function.
Exercise 5: Let $y \in f(A)$. Then there is an $x \in A$ such that $f(x) = y$. As $A \subset B$, $x \in B$, so $y \in f(B)$. Thus $f(A) \subset f(B)$.
Exercise 6: Let $f : \mathbb{R} \to \mathbb{R}$, with $f(x) = 0$. Let $A = [0, 1]$ and $B = [2, 3]$. Then $A \cap B = \emptyset$, so $f(A \cap B) = \emptyset$. However, we have $f(A) = f(B) = \{0\}$, so $f(A) \cap f(B) = \{0\}$.
Exercise 7: We just need to show the other inclusion. Let $y \in f(A) \cap f(B)$. Then $y \in f(A)$ and $y \in f(B)$, so there is an $a \in A$ and $b \in B$ so that $f(a) = f(b) = y$. But as f is injective, we must have $a = b$. Thus there is a unique x so that $f(x) = y$, and $x \in A$ and $x \in B$, so $x \in A \cap B$. Thus $y \in f(A \cap B)$.
Exercise 8: Let $y \in f(A \cup B)$. Then there exists an $x \in A \cup B$ such that

$f(x) = y$. There are two cases. First, suppose $x \in A$. Then $y \in f(A)$, so $y \in f(A) \cup f(B)$. If instead $x \in B, y \in f(B)$, so $y \in f(A) \cup f(B)$. Thus $f(A \cup B) \subset f(A) \cup f(B)$.
Now suppose $y \in f(A) \cup f(B)$. Again, there are two cases. Suppose $y \in f(A)$. Then there is an $x \in A \subset A \cup B$ such that $f(x) = y$, so $y \in f(A \cup B)$. The other case is similar. Thus $f(A) \cup f(B) \subset f(A \cup B)$. And so we have equality.
Exercise 9: Let $y \in f(A) \setminus f(B)$. Then $y \in f(A)$, so there is an $x \in A$ such that $f(x) = y$. As $y \notin f(B)$, there is no $x \in B$ such that $f(x) = y$. Thus $x \in A \setminus B$, and so $y \in f(A \setminus B)$.
Exercise 10: Let $f : \mathbb{N} \to \mathbb{N}$, where $f(x) = x^2$. Let $A = \{-1, 0, 1\}$ and $B = \{-1\}$. Then $f(A \setminus B) = f(\{0, 1\}) = \{0, 1\}$. We have $f(A) = \{0, 1\}$ and $f(B) = \{1\}$, so $f(A) \setminus f(B) = \{0\}$.
Exercise 11: We just need the other set inclusion. Suppose $y \in f(A \setminus B)$. Then there is an $x \in A \setminus B$ such that $f(x) = y$. As $x \in A$, $y \in f(A)$. To show $y \notin f(B)$, suppose there were a $z \in B$ such that $f(z) = y$. Then $f(x) = f(z)$, so $x = z$. But as $z \in B, x \in B$, contradiction. Thus there is no $z \in B$ such that $f(z) = y$, and we have $y \notin f(B)$. Thus $y \in f(A) \setminus f(B)$.
Exercise 12: We have $f(A) \triangle f(B) = (f(A) \cup f(B)) \setminus (f(A) \cap f(B)) = f(A \cup B) \setminus (f(A) \cap f(B)) \subset f(A \cup B) \setminus f(A \cap B) \subset f((A \cup B) \setminus (A \cap B)) = f(A \triangle B)$, where the first inclusion came from exercise 6 and second from exercise 9.
If $B \subset A, A \triangle B = A \setminus B$. Thus the example from exercise 10 will work here too.
Lastly, if f is injective, both of the set inclusions from above will be equalities from exercise 7 and 11.
Exercise 13:

$$f(n) = \begin{cases} \frac{1}{2}n & \text{if } n \text{ even} \\ -\frac{1}{2}(n-1) & \text{if } n \text{ odd} \end{cases}$$

Exercise 14: Let c be any fixed element of X. Let $y \in Y$. Then either $y \in f(X)$ or $y \notin f(X)$. If $y \in f(X)$, then there is only one $x \in X$ such that $f(x) = y$. So define g such that $g(y) = x$. If $y \notin f(X)$, then let $g(y) = c$. Now we have g defined everywhere on Y, and every element of X is the image of some element of Y by g. To recap, we have:

$$g(y) = \begin{cases} x \text{ so that } f(x) = y & \text{if } y \in f(X) \\ c & \text{if } y \notin f(X) \end{cases}$$

Exercise 15: Let $y \in Y$. Look at the set $f^{-1}(y) = \{x \in X | f(x) = y\}$. For each $y \in Y$, pick one element (call it x) out of the corresponding set (to do this, we technically need something called the Axiom of Choice, but that is beyond our scope here). Define g so that $g(y) = x$. Then g is injective. To

recap, we have: $g(y) = x$, for any fixed x such that $f(x) = y$.
****Exercise 16 (Schröder-Bernstein Theorem)**: I will take my own advice and run away (a nice proof can be found on Wikipedia).