Summer Session Week 4

Notes

July 2021

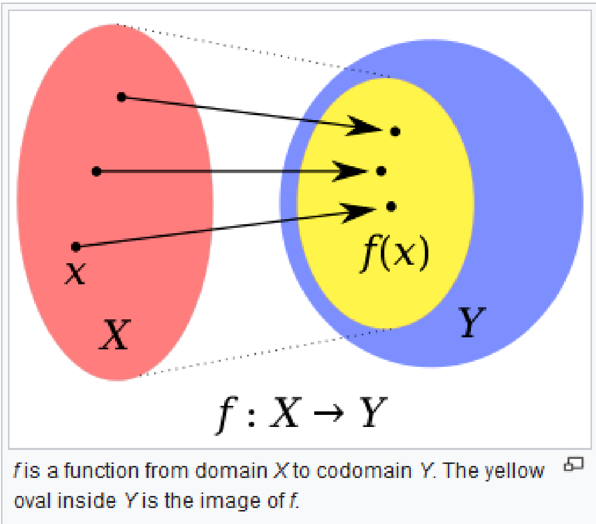

Figure 1: Visualization of a function (wikipedia)

- There are a few ways to define a function. For us, we will say a function is a way to move points from one set to points in another set.
- Notation: Let f be the function moving points from a set X to a set Y. Then we write $f : X \to Y$. This is read as "f is a function from X into Y". Sometimes we use the word "map" instead of "function".
- The set X is called the *domain* and the set Y is called the *codomain*.
- To show how the function moves points, we usually write the function explicitly. If $x \in X$ and $y \in Y$, we write $f(x) = y$. We call x the *argument* of the function.

- Example: Let $f : \mathbb{N} \to \mathbb{N}$, and $f(x) = 2x$. What this function does is it takes any natural number, and doubles it.

- Example: Let $f : \mathbb{N} \setminus \{1, 2\} \to S$, where S is the set of regular shapes. What f will do is take a number, and send that number to the shape with that many vertices. For example, $f(3) = \triangle, f(4) = \square, f(5) = ⬠$.

- Example: Let C be the set of countries, W be the set of fictional writers, and M be the set of mathematicians. Let $f : C \to W \times M$, where f takes in a country, and outputs an ordered pair of the most famous fictional writer and mathematician born in that country. For example, $f(\text{USA}) = (\text{Ernest Hemingway}, \text{Michael Freedman})$. What do you think $f(\text{Russia})$ is? How about $f(\text{Scotland})$?

- Example (Stars Over Babylon): Let $f : \mathbb{R} \cap [0, 1] \to \mathbb{R}$. The domain is all of the real numbers between 0 and 1. What will this function do? If $x \in \mathbb{R} \setminus \mathbb{Q}$ (irrational), then let $f(x) = 0$. If $x \in \mathbb{Q}$, then write $x = \frac{p}{q}$, with the fraction fully reduced. Let $f(\frac{p}{q}) = \frac{1}{q}$. We have now described what f does to all points in its domain. We can write this function is a neat way using piecewise notation:

$$f(x) = \begin{cases} 0 & \text{x is irrational} \\ \frac{1}{q} & x = \frac{p}{q} \end{cases}$$

- Image: Let A be a subset of our domain, $A \subset X$. Then the *image of A* is the set of everything A gets sent to. This is written as $f(A)$. Note here that the argument is a set, instead of just a point. Using set builder notation, we have $f(A) = \{f(x) \in Y | x \in A\}$. The *image of f* is the image of the domain, $f(X)$.

- Example: Looking back at our first function, if $A = \{1, 2, 3\}$, then $f(A) = \{2, 4, 6\}$.

- Surjection (onto): A function f is called *surjective*, or we say f is a *surjection*, if the image of the domain is equal to the codomain; $f(X) = Y$ (we always have $f(X) \subset Y$). Another way to say this is to say that f is an *onto* function. In this case, instead of saying "f is a map from X *into* Y", we would say "f is a map from X *onto* Y".

- Injection (one-to-one): A function f is called *injective*, or we say f is an *injection*, if no two points in X gets sent to the same point in Y. To prove a function is injective, assume $f(a) = f(b)$, then prove that

$a = b$. By contrapositive, this is the same as saying if $a \neq b$, then $f(a) \neq f(b)$. Another way to say this is to say that f is a *one-to-one* function, or that f is one-to-one.

- Bijection (one-to-one correspondence): A function f is called *bijective*, or we say f is a *bijection*, if f is both injective and surjective. Another way to say this is to say that f is a one-to-one correspondence. That is, f takes points from X, and maps it to points to Y, such that each point in X is sent to a unique point in Y, and every point in Y is the image of some point in X.

- Example: Looking back at our first function, f is not surjective, as if there were an $x \in \mathbb{N}$ such that $f(x) = 3$, then $x = \frac{3}{2}$. But $\frac{3}{2} \notin \mathbb{N}$. However, f is injective. Assume $f(a) = f(b)$. Then $2a = 2b$, and so $a = b$.

- Example: Looking at our second example, f is not injective. This is because $f(\frac{1}{3}) = f(\frac{2}{3})$, but $\frac{1}{3} \neq \frac{2}{3}$. Our function is also not surjective. But if we changed the codomain from $\mathbb{R}$ to $\{x \in \mathbb{Q} | x = \frac{1}{n}$ for some $n \in \mathbb{N}$ or $x = 0\}$, then f would be surjective. In fact, all functions are surjective if we change the codomain to be the image of f.

- Example proof: Let $A, B \subset X$. We will prove $f(A \cap B) \subset f(A) \cap f(B)$. Let $y \in f(A \cap B)$. Then there is an $x \in A \cap B$ such that $f(x) = y$. As $x \in A, y \in f(A)$. As $x \in B, y \in f(B)$. Thus $y \in f(A) \cap f(B)$, so $f(A \cap B) \subset f(A) \cap f(B)$.

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Exercises

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Let $f_1 : \mathbb{Z} \to \mathbb{Z}, f_2 : \mathbb{N} \to \mathbb{Z}, f_3 : \mathbb{Z} \to \mathbb{N}, f_4 : \mathbb{N} \to \mathbb{N}$, and $f(x) = x^2$ is the rule for all of these functions.

Exercise 1: Are any of these not real functions?
Exercise 2: Are any of these functions injective? Surjective? Bijective?
Exercise 3: What is $f_1(\{-3, -1, 1, 3\})$?
Exercise 4: Let $f : \mathbb{R} \to \mathbb{R}$. Prove that f in bijective, where

$$f(x) = \begin{cases} 2x - 3 & x < 1 \\ 5x - 6 & x \geqslant 1 \end{cases}$$

(Hint: for injective, split into cases depending on if $x < 1$ or $x \geqslant 1$. For surjective, split into cases depending on if $y < -1$ or if $y \geqslant -1$.)

Let $f : X \to Y$ be an arbitrary function between two arbitrary sets.

Exercise 5: Suppose $A \subset B \subset X$. Prove $f(A) \subset f(B)$.
Exercise 6: We have proved that, if $A, B \subset X$, then $f(A \cap B) \subset f(A) \cap f(B)$. Find a function f and sets A, B, X, Y such that $f(A \cap B) \neq f(A) \cap f(B)$.
Exercise 7: Suppose, in the context of the previous problem, that f is injective. Prove that $f(A \cap B) = f(A) \cap f(B)$.
Exercise 8: Suppose $A, B \subset X$. Prove $f(A \cup B) = f(A) \cup f(B)$.
Exercise 9: Suppose $A, B \subset X$. Prove that $f(A \setminus B) \supset f(A) \setminus f(B)$.
Exercise 10: In the context of the previous problem, show that it is possible to have $f(A \setminus B) \neq f(A) \setminus f(B)$.
Exercise 11: In the context of the previous two problems, suppose furthermore that f is injective. Prove then that $f(A \setminus B) = f(A) \setminus f(B)$.
Exercise 12: Repeat exercises 8-10, replacing $\setminus$ with $\triangle$.
Exercise 13: Come up with a bijection between $\mathbb{N}$ and $\mathbb{Z}$

Exercise 14: Suppose $f : X \to Y$ is injective. Prove that there exists a surjective function $g : Y \to X$.
Exercise 15: Suppose $f : X \to Y$ is surjective. Prove that there exists a injective function $g : Y \to X$.
***__Exercise 16 (Schröder-Bernstein Theorem)__: Suppose there exists an injection $f : X \to Y$ and there exists and injection $g : Y \to X$. Prove there exists a bijection between X and Y.