Summer Session Week 3

Solutions

July 2021

Exercise 1: $A \times B = \{\{1,a\},\{1,b\},\{1,c\},\{2,a\},\{2,b\},\{2,c\}\}$
Exercise 2: $\emptyset, A \subset A$
Exercise 3: There should be $2^3 = 8$ elements (see exercise 12). $\mathcal{P}(A) = \{\emptyset,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$
Exercise 4: Let $(a,b) \in A\times(B\cap C)$. Then $a \in A$ and $b \in B\cap C$, so $b \in B$ and $b \in C$. Thus $(a,b) \in A \times B$ and $(a,b) \in A \times C$, so $(a,b) \in (A \times B) \cap (A \times C)$. So $A \times (B \cap C) \subset (A \times B) \cap (A \times C)$.
Now let $(a,b) \in (A\times B)\cap(A\times C)$, so that $(a,b) \in A\times B$ and $(a,b) \in A\times C$. From the first one, $a \in A$ and $b \in B$, and the second one gives us $b \in C$. Thus $b \in B \cap C$, so $(a,b) \in A \times (B \cap C)$. Thus $(A \times B) \cap (A \times C) \subset A \times (B \cap C)$. Hence equality.
Exercise 5: Let $(a,b) \in A \times (B \cup C)$. Then $a \in A$ and $b \in B \cup C$, so $b \in B$ or $b \in C$. If $b \in B$, then $(a,b) \in A \times B$, so $(a,b) \in (A \times B) \cup (A \times C)$. If $b \in C$, then $(a,b) \in A \times C$, so $(a,b) \in (A \times B) \cup (A \times C)$. As these are all the cases, we have $A \times (B \cup C) \subset (A \times B) \cup (A \times C)$.
Now suppose $(a,b) \in (A\times B)\cup(A\times C)$. Then $(a,b) \in A\times B$ or $(a,b) \in A\times C$. In the first case, $a \in A$ and $b \in B$, so $b \in B \cup C$. In the second case, $a \in A$ and $b \in C$, so $b \in B \cup C$. In either case, $a \in A$ and $b \in B \cup C$, so $(a,b) \in A\times(B\cup C)$. Thus $(A\times B)\cup(A\times C) \subset A\times(B\cup C)$. Hence equality.
Exercise 6: Let $(a,b) \in A \times (B \setminus C)$. Then $a \in A$ and $b \in B \setminus C$, so $b \in B$ and $b \notin C$. As $b \in B$, $(a,b) \in A \times B$. As $b \notin C$, $(a,b) \notin A \times C$. Thus $(a,b) \in (A \times B) \setminus (A \times C)$, and so $A \times (B \setminus C) \subset (A \times B) \setminus (A \times C)$.
Now suppose $(a,b) \in (A \times B) \setminus (A \times C)$. Then $(a,b) \in A \times B$, so $a \in A$ and $b \in B$. Also, $(a,b) \notin A \times C$, so either $a \notin A$ or $b \notin C$. As $a \in A$, we must have $b \notin C$. Thus $b \in B \setminus C$, so $(a,b) \in A \times (B \setminus C)$. And so $(A \times B) \setminus (A \times C) \subset A \times (B \setminus C)$, thus we have equality.
Exercise 7: Let $(a,c) \in (A\cap B) \times (C \cap D)$. Then $a \in A\cap B$ and $c \in C \cap D$. Thus $a \in A$ and $c \in C$, so $(a,b) \in A \times C$. Also, $a \in B$ and $c \in D$, so $(a,c) \in B \times D$. So $(a,c) \in (A \times C) \cap (B \times D)$. Thus $(A \cap B) \times (C \cap D) \subset$

$(A \times C) \cap (B \times D)$.
Now let $(a, c) \in (A \times C) \cap (B \times D)$. Then $(a, c) \in A \times C$, so $a \in A$ and $c \in C$. Also, $(a, c) \in B \times D$, so $a \in B$ and $c \in D$. Thus $a \in A \cap B$ and $c \in C \cap D$, so $(a, b) \in (A \cap B) \times (C \cap D)$. Hence $(A \times C) \cap (B \times D) \subset (A \cap B) \times (C \cap D)$, and we have equality.
Exercise 8: Suppose $A \subset B$, and let $(a, c) \in A \times C$. Then $a \in A$, and so $a \in B$. As $c \in C$, we have $(a, c) \in B \times C$, and so $A \times C \subset B \times C$.
Exercise 9: First suppose $A \times B \subset C \times D$. Suppose $a \in A$ and $b \in B$. Then $(a, b) \in A \times B$, so $(a, b) \in C \times D$. Thus $a \in C$ and $b \in D$. Hence $A \subset C$ and $B \subset D$.
Now suppose $A \subset C$ and $B \subset D$. Let $(a, b) \in A \times B$. Then $a \in A$, so $a \in C$. Also, $b \in B$, so $b \in D$. Thus $(a, b) \in C \times D$, and so $A \times B \subset C \times D$.
Exercise 10: Suppose $|A| = n$ and $|B| = m$, and write $A = \{x_1, \ldots, x_n\}$. Then $A \times B = \{x_1\} \times B \cup \ldots \cup \{x_n\} \times B$, and each of the sets in the union are disjoint as each x_i is unique. Each set in the union also has m elements, as there is only one choice for the first coordinate, and m for the second. So $|A \times B| = m + \ldots + m, n$ times, which is $n \cdot m = |A| \cdot |B|$.
Exercise 11: The previous exercise is the base case. Now suppose equality holds for n sets, and let X_{n+1} be finite. Then, again by the previous exercise, $|X_1 \times \ldots X_n \times X_{n+1}| = |(X_1 \times \ldots X_n) \times X_{n+1}| = |(X_1 \times \ldots X_n)| \cdot |X_{n+1}| = |X_1| \cdot \ldots \cdot |X_n| \cdot |X_{n+1}|$.
***Exercise 12**: Suppose $|A| = n$, and form a bit string of length n (this is just a list of n 0's or 1's). List the elements of A in any order, so that $A = \{x_1, \ldots x_n\}$. For each element in the power set, we can correspond that to exactly one choice of bit string. How do we do this? Take an element of the power set. If x_i is in the set, then set the ith bit to 1, and if x_i is not in the set, set the ith bit to 0. Every subset of A is then corresponded to a unique bit string of length n, and each bit string of length n is corresponded to a subset of A. Counting the amount of different bit strings is now just counting to $1 \ldots 1$, n 1's, in binary, plus the string $0 \ldots 0$, n 0's. This is exactly $(2^n - 1) + 1$.
***Exercise 13**: Step 1: The set of everything that is not a teacup is not a teacup, so that set is in itself.
Step 2: Defined.
Step 3: Suppose $R \in R$. Then, by the definition of R, $R \notin R$, contradiction. So $R \notin R$. But if $R \notin R$, as $R \in \Omega$, by the definition of R, $R \in R$. Contradiction, so $R \in R$. But we already proved that to be impossible.
Step 4: By step 3, R cannot exist. There are in fact two places we could have gone wrong. First, that the set Ω exists. If $R \notin \Omega$ (which is impossible if Ω is the set of all sets), we would be fine, as then $R \notin R$. The second place we could be wrong is a bit more subtle, and that is that the set R may not exist.

We have assumed that any set of the form $\{x \in \Omega | P(x)\}$, where $P(x)$ is a proposition, is a real set. This is called the *Axiom Schema of Specification.* So if you would like there to be a set of all sets, you would have to build set theory without this axiom schema.