Summer Session Week 3

Notes

July 2021

- There are a couple ways you may see sets being written with set builder notation that are not rigorous, but accepted. One way is the following way to write the set of all square numbers: $\{n^2 | n \in \mathbb{N}\}$. This technically should be written as $\{n \in \mathbb{N} | n = m^2 \text{ for some } m \in \mathbb{N}\}$. For the sake of practice, you should only use the second way of writing sets.

- Another thing that is often done is the author may just drop the universal set. When this is done, it will always be clear from context what the universe is. For example, $\{x | x \text{ is prime}\}$. We are talking about prime numbers, so this really only makes sense when the universe is $\mathbb{N}$ or $\mathbb{Z}$. This set should be written as $\{x \in \mathbb{N} | x \text{ is prime}\}$, but it is generally accepted to use shortcuts. But as we are just learning this topic, we will take no shortcuts and be completely rigorous.

- Power Set: The power set of a set A is written as $\mathcal{P}(A)$. This is the set of all subsets of A. $\mathcal{P}(A) = \{x \in \mathcal{P}(\Omega) | x \subset A\}$. For example, if $A = \{1, 2\}$, then $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.

- For our next definition, we first need the concept of ordered pairs. An ordered pair is a 2-tuple, written (a, b). This is called an *ordered* pair as order matters: (a, b) may be different from (b, a). We can further have ordered n-tuples: $(x_1, \ldots, x_n)$. Again, order matters. In fact, $(a_1, b_1) = (a_2, b_2)$ if and only if $a_1 = a_2$ and $b_1 = b_2$. The interested reader may consider how to write ordered pairs using sets only. One way is to say $(a, b) = \{\{a\}, \{a, b\}\}$. There are other definitions, but this is beyond our scope of interest here.

- When talking about the set A, let Ω_A be the universe that contains A.

- Cartesian Product: The Cartesian product (also called product) of two sets A and B is written as $A \times B$. This is the set of all ordered pairs with the first coordinate in A and the second coordinate in B, i.e. $A \times B = \{(a, b) \in \Omega_A \times \Omega_B | a \in A \text{ and } b \in B\}$. We usually write $A \times A$ as A^2.

- We can also have the product of many sets. $X_1 \times X_2 \times \ldots \times X_n = \{(x_1, \ldots, x_n) \in \Omega_{X_1} \times \ldots \Omega_{X_n} | x_i \in X_i \text{ for all } i\}$.

- For reasons beyond the scope of this class, $(A \times B) \times C$, $A \times (B \times C)$, and $A \times B \times C$ are all regarded as the same.

- Example: $\mathbb{R}$ is the real line, $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ is the plane.

- Example: $[0, 1]$ is a line segment, $[0, 1] \times [0, 1]$ is a square with vertices at $(0, 0), (0, 1), (1, 0), (1, 1)$. Here, the ordered pairs are regarded as coordinates in the plane.

- Example: Let S^1 be the unit circle. Then $S^1 \times S^1$ is a hollow doughnut, also called the torus. If B^2 is is the unit disk, a filled in unit circle, then $B^2 \times S^1$ is a thick doughnut.

- We will talk more about this later, but for finite sets, the number of elements is called the cardinality of the set. If A is a set, the cardinality of A is written $|A|$.

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Exercises

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If a problem has a *, that means it is difficult. I will usually give hints on how to do it. Problems with ** are extra difficult. If a problem has ***, run.

Exercise 1: Let $A = \{1, 2\}$, and $B = \{a, b, c\}$. What is $A \times B$?
Exercise 2: Why is $\emptyset, A \in \mathcal{P}(A)$?
Exercise 3: What is $\mathcal{P}(A)$, when $A = \{1, 2, 3\}$? Before you do this, how many elements do you think should be in $\mathcal{P}(A)$?
Exercise 4: Prove the distribution law: $A \times (B \cap C) = (A \times B) \cap (A \times C)$.
Exercise 5: Prove the distribution law: $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
Exercise 6: Prove the distribution law: $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$.
Exercise 7: Prove $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$.
Exercise 8: Prove that if $A \subset B$, then $A \times C \subset B \times C$.
Exercise 9: Prove $A \times B \subset C \times D$ if and only if $A \subset C$ and $B \subset D$.
Exercise 10: For now, take for granted that if A and B are disjoint ($A \cap B = \emptyset$), then $|A \cup B| = |A| + |B|$ (we will prove this later). Prove that, if A and B are finite, then $|A \times B| = |A| \cdot |B|$.
Exercise 11: If you know induction, use induction, plus the previous exercise, to prove that, if X_i is finite for all i, then $|X_1 \times \ldots X_n| = |X_1| \cdot \ldots \cdot |X_n|$.
***Exercise 12**: If $|A| = n$, prove $\mathcal{P}(A) = 2^n$.
***Exercise 13**: This problem uses what is known as "Russell's paradox". Last time, we talked a bit about a universal set. We noted that it changes depending on context. Why can't there be just one huge universal set that always works? This would be the set of all sets. We prove that this set cannot exists (at least, in the way we have developed set theory thus far).
Step 1: Is it possible for a set to be a member of itself? Think of the set of everything that is a teacup. Then think of the set of everything which is not a teacup.
Step 2: Suppose that Ω was a set of all sets. Define a new set R, a set of sets, such that $R = \{x \in \Omega | x \notin x\}$. Notice here that the x's are sets.

Step 3: Consider the question "Is $R \in R$? Is $R \notin R$?"
Step 4: Conclude that the set R cannot exist, contradicting the fact that R should exist. Where did we go wrong?