Modular Arithmetic (General Problems) Solutions

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1. If a biology experiment begins at 7:00AM and runs for 80 hours, at what time will it end?

   Solution

   Since there are 24 hours a day, we can compute how many days are in 80 days. We see that there are 80/24 days in 80 hours, or $3\frac{8}{24}$ days in 80 hours. We know that the number of days does not affect the time of day the biology experiment will end, so we really only care about the remainder of hours divided by 24, which is 8hrs past 7:00AM, or 3:00PM.

2. Cory's birthday lies on a Monday this year. What day of the week will his birthday be on in 2022?

   (There was a typo with the original problem, it should have been 2022).

   Solution

   There are about 365 days in a year—we suppose that there are precisely this many days in a year. We want to know what day of the week his birthday will be on the next year. Note that there are 7 days in a week, so we are really asking what is the remainder of 365 divided by 7 because that will give us the number of days past a cycle of 7 days in a week. We compute this remainder and find that there are 52 weeks in a year plus one day. Then, a day after Monday from the previous year would be Tuesday.

3. Reduce the following numbers using modular arithmetic:

   (a) $136283 \times 192758237582389 \equiv \pmod 2$

   Solution

   When we take mod 2 of any number we are really asking if the number is odd or even. We don't even have to compute this strange product, we can take a look at the ones place and take the product of 3 and 9, which is 27, and see that 7 is an odd number in the ones place in the product of these two numbers. This product is congruent to 1 mod 2.

   (b) $19342347328 + 1894837483 \equiv \pmod{10}$

   Solution

   Similar to the above, when we take mod 10, we are simply extracting the ones place number in this sum. Then, taking the sum of 3 and 8, we get 11, the ones place of 11 is just 1, so the sum of these numbers is congruent to 1 mod 10.

   (c) $1934232 \times 1894837480 \equiv \pmod{10}$

   Solution

   This problem is very much the same as the above, instead we take products of numbers in the ones place, between 0 and 2, which is 0, and find that the product of these two numbers mod 10, is just 0. So the product of these numbers is congruent to 0 mod 10.

(d) Suppose hot dog buns come in packages of 34, and hot dogs come in packages of 8.

i. What is the smallest number of packages of hot dogs and hot dog buns Ivy should buy if she doesn't want to have left-over hot dogs or left-over hot dog buns? (Assume that hot dogs can't be eaten without a bun, or vice versa).

Solution

Since we are asking how many packages of buns and hot dogs we need to buy so that there are no left overs, we can also interpret this problem as what is smallest number of hot dogs and buns we can have. Further, we are really asking what is the lowest common multiple of 34 and 8? We can take a look at their prime factorization

$$34 = 2 \times 17 \quad \text{and} \quad 8 = 2 \times 2 \times 2$$

Between both numbers we have common factors of one 2. The uncommon factors between both are two 2's missing from 24, and 17 missing from 8. The lowest common multiple however is really 17×8 or 34×4. Therefore, we should buy 4 packages of hot dog buns and 8 packages of hot dogs.

ii. Suppose that hot dog buns come in packages of 33. What is the smallest number of packages of hot dogs and hot dog buns Ivy should buy now?

Solution

We use the same approach for 33 instead. We see here however, that in the prime factorization of 33, there are no common factors between

$$33 = 11 \times 3 \quad \text{and} \quad 8 = 2 \times 2 \times 2.$$

The lowest common multiple in this case is exactly the product of 33 and 8, or 264. The number of hot dog bun packages we should buy is 8, and the number of hot dog packages we should buy is 33.

iii. Now assume hot dog buns come in packages of n. Write expressions that show how many packages of hot dog buns Ivy should buy. Note that there will be two expressions: one where the reduced form of n in mod 8 is divisible by 8, and one where it is not.

Solution

The problems immediately above can be reduced to the following. What is n mod 8? In the first problem, 34 mod 8 is 2. We had to purchase 4 packages of hot dog buns so that this remainder would not occur, as in $(34 \bmod 8) \times 4 = 2 \times 4 = 8$, which is divisible by 8. Then, the required expression for the number of hot dog packages we should buy is

$$\frac{8}{n \bmod 8}, \quad \text{if } (n \bmod 8) \text{ divides 8, with no remainder.}$$

Otherwise, like in the problem immediately above when n mod 8 does not divide 8, we can simply purchase 8 packages of hot dog buns.