Summer Session Week 1

Solutions

June 2021

Exercise 1: $\{2\}$
Exercise 2: $\{-2, 2\}$
Exercise 3: $\emptyset$
Exercise 4: $\{-i, i\}$
Exercise 5: $A \cup B = \{1, 2, 3, 4, 5, 6\}, A \cap B = \{3, 4\}, A \setminus B = \{1, 2\}, B \setminus A = \{5, 6\}, A \triangle B = \{1, 2, 5, 6\}$
Exercise 6: Suppose $\emptyset \not\subseteq A$. Then there is an element $x \in \emptyset$ such that $x \notin A$. But this is impossible as $\emptyset$ has no elements.
Exercise 7: Yes, yes, yes, no, yes
Exercise 8: A
Exercise 9: $\emptyset$
Exercise 10: $\{x \in \mathbb{Z} | x \leq 0\} = \{0, -1, -2, \ldots\}$
Exercise 11: We need to find an element $x \in \mathbb{Z}$ such that $x \notin \mathbb{N}$. $x = -1$ works. For the second part, $x = i$ works.
Exercise 12: Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. So, in particular, $x \in B$. So $A \cap B \subset B$.
Exercise 13: Suppose $B \subset A$. Let $x \in B$. Then $x \in A$, so $x \in A \cap B$. So $B \subset A \cap B$. By previous exercise, we then have $A \cap B = B$.
Now suppose $A \cap B = B$. Let $x \in B$. Then $x \in A \cap B (= B)$, so $x \in A$. So $B \subset A$.
Exercise 14: Let $x \in A$. Then $x \in A$ or $x \in B$, so $x \in A \cup B$. So $A \subset A \cup B$.
Exercise 15: Suppose $B \subset A$. Let $x \in A \cup B$. Then $x \in A$ or $x \in B$. If $x \in B$, then $x \in A$. The other case also has $x \in A$. So in both cases, $x \in A$, so $A \cup B \subset A$. By previous exercise, $A \cup B = A$.
Now suppose $A \cup B = A$. Let $x \in B$. Then $x \in A \cup B = A$, so $x \in A$, and we have $B \subset A$.
Exercise 16: Let $x \in (A \cup B) \cup C$. Then $x \in (A \cup B)$ or $x \in C$. If $x \in C$, then $x \in B \cup C$, and also $x \in A \cup (B \cup C)$. The other case is $x \in A \cup B$. Then $x \in A$ or $x \in B$. If $x \in A$, $x \in A \cup (B \cup C)$. If $x \in B$, then $x \in B \cup C$,

and $x \in A \cup (B \cup C)$. Hence, no matter what, we have $x \in A \cup (B \cup C)$. So $(A \cup B) \cup C \subset A \cup (B \cup C)$. The other direction is similar.

Exercise 17: Let $x \in (A \cap B) \cap C$. Then $x \in C$ and $x \in A \cap B$, so $x \in A$ and $x \in B$. In particular, $x \in B$ and $x \in C$, so $x \in B \cap C$. As $x \in A$ too, we have $x \in A \cap (B \cap C)$. Hence $(A \cap B) \cap C \subset A \cap (B \cap C)$. Other direction is similar.

Exercise 18: Let $x \in (A \cup B)^{\mathsf{c}}$. Then $x \notin A \cup B$. So we must have $x \notin A$ and $x \notin B$. So $x \in A^{\mathsf{c}}$ and $x \in B^{\mathsf{c}}$, so $x \in A^{\mathsf{c}} \cap B^{\mathsf{c}}$. So $(A \cup B)^{\mathsf{c}} \subset A^{\mathsf{c}} \cap B^{\mathsf{c}}$. Now let $x \in A^{\mathsf{c}} \cap B^{\mathsf{c}}$. Then $x \in A^{\mathsf{c}}$ and $x \in B^{\mathsf{c}}$. So $x \notin A$ and $x \notin B$. So $x \notin A \cup B$, and $x \in (A \cup B)^{\mathsf{c}}$. So $A^{\mathsf{c}} \cap B^{\mathsf{c}} \subset (A \cup B)^{\mathsf{c}}$. Hence we have equality.

Exercise 19: Let $x \in (A \cap B)^{\mathsf{c}}$. Then $x \notin A \cap B$, so $x \notin A$ or $x \notin B$. So either $x \notin A$ or $x \notin B$. In either case, $x \in A^{\mathsf{c}} \cup B^{\mathsf{c}}$, so $(A \cap B)^{\mathsf{c}} \subset A^{\mathsf{c}} \cup B^{\mathsf{c}}$. Now let $x \in A^{\mathsf{c}} \cup B^{\mathsf{c}}$. If $x \in A^{\mathsf{c}}$, then $x \notin A \supset A \cap B$, so $x \notin A \cap B$, and $x \in (A \cap B)^{\mathsf{c}}$. Similar if $x \in B^{\mathsf{c}}$. So $A^{\mathsf{c}} \cup B^{\mathsf{c}} \subset (A \cap B)^{\mathsf{c}}$. Hence we have equality.

Exercise 20: $(A \cup B \cup C)^{\mathsf{c}} = ((A \cup B) \cup C)^{\mathsf{c}} = (A \cup B)^{\mathsf{c}} \cap C^{\mathsf{c}} = A^{\mathsf{c}} \cap B^{\mathsf{c}} \cap C^{\mathsf{c}}$

Exercise 21: Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. If $x \in B$, then $x \in A \cap B$, so $x \in (A \cap B) \cup (A \cap C)$. If $x \in C$, then $x \in A \cap B$, so $x \in (A \cap B) \cup (A \cap C)$. Hence $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$.

Now suppose $x \in (A \cap B) \cup (A \cap C)$. If $x \in A \cap B$, then $x \in A$, and $x \in B \subset B \cup C$, so $x \in A \cap (B \cup C)$. If $x \in A \cap C$, then $x \in A$ and $x \in C \subset B \cup C$, so $x \in A \cap (B \cup C)$. So $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$. Hence equality.

Exercise 22: Let $x \in A \cup (B \cap C)$. If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$, so $x \in (A \cup B) \cap (A \cup C)$. If $x \in B \cap C$, then $x \in B \subset A \cup B$ and $x \in C \subset A \cup C$, so $x \in (A \cup B) \cap (A \cup C)$. Hence $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$.

Now suppose $x \in (A \cup B) \cap (A \cup C)$. Then $x \in A \cup B$ and $x \in A \cup C$. In the first one, suppose $x \in A$. Then $x \in A \cup (B \cap C)$. The other case is $x \in B$. So suppose $x \in B$. Then from the second, either $x \in A$ or $x \in C$. If $x \in A$, then $x \in A \cup (B \cap C)$. If not, then $x \in C$. So $x \in B \cap C$, so $x \in A \cup (B \cap C)$. Therefore $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$. Hence equality.

Exercise 23: Let $x \in (A^{\mathsf{c}})^{\mathsf{c}}$. Then $x \notin A^{\mathsf{c}}$, so $x \in A$. So $(A^{\mathsf{c}})^{\mathsf{c}} \subset A$. Let $x \in A$. Then $x \notin A^{\mathsf{c}}$, so $x \in (A^{\mathsf{c}})^{\mathsf{c}}$, so $A \subset (A^{\mathsf{c}})^{\mathsf{c}}$, hence equality.

Exercise 24: Let $x \in A \setminus B$. Then $x \in A$ and $x \notin B$. So $x \in B^{\mathsf{c}}$. So $x \in A \cap B^{\mathsf{c}}$, and $A \setminus B \subset A \cap B^{\mathsf{c}}$.

Now let $x \in A \cap B^{\mathsf{c}}$. Then $x \in A$ and $x \notin B$, so $x \in A \setminus B$. So $A \cap B^{\mathsf{c}} \subset A \setminus B$. Hence we have equality.

Exercise 25: The fastest way to do this is to use De Morgan's rule and the previous exercise. $(A \setminus B)^{\mathsf{c}} = (A \cap B^{\mathsf{c}})^{\mathsf{c}} = A^{\mathsf{c}} \cup (B^{\mathsf{c}})^{\mathsf{c}} = A^{\mathsf{c}} \cup B$.

Exercise 26: A

Exercise 27: $\emptyset$
Exercise 28: Ω (notice how this depends on which universe we are using)
Exercise 29: Suppose $A \subset B$. Let $x \in B^{\mathsf{c}}$. Suppose $x \in A$. Then $x \in B$, but that contradicts the fact that $x \in B^{\mathsf{c}}$. So $x \notin A$, so $x \in A^{\mathsf{c}}$, and $B^{\mathsf{c}} \subset A^{\mathsf{c}}$. Now suppose $B^{\mathsf{c}} \subset A^{\mathsf{c}}$. Let $x \in A$. Suppose $x \notin B$, so that $x \in B^{\mathsf{c}}$. Then $x \in A^{\mathsf{c}}$, so $x \notin A$, which is a contradiction. So $x \in B$, and $A \subset B$.
Exercise 30: $A = \{1, 2\}, B = \{2, 3\}$
Exercise 31: This is basically the definition of $A \triangle B$.
Exercise 32: We will use the previous exercise's definition of $A \triangle B$. Suppose $x \in (A \setminus B) \cup (B \setminus A)$. One case is that $x \in A \setminus B$. In this case, $x \in A$ and $x \notin B$. As $x \in A$, $x \in A \cup B$. As $A \cap B \subset B$, and $x \notin B$, we have $x \notin A \cap B$. So $x \in (A \cup B) \setminus (A \cap B)$. The other case is similar. Thus $(A \setminus B) \cup (B \setminus A) \subset (A \cup B) \setminus (A \cap B)$.
Now suppose $x \in (A \cup B) \setminus (A \cap B)$. Then $x \in A \cup B$ and $x \notin A \cap B$. So $x \notin A$ or $x \notin B$. Suppose $x \notin A$. By $x \in A \cup B$, we must have $x \in B$ (else if $x \in A$, contradiction). So $x \in B \setminus A$, and $x \in (A \setminus B) \cup (B \setminus A)$. The case where $x \notin B$ is similar. Hence $(A \cup B) \setminus (A \cap B) \subset (A \setminus B) \cup (B \setminus A)$. Therefore we have equality.