Summer Session Week 1

Notes

June 2021

- A set is a collection of things, written in curly braces: {cat, dog, 4, π}. Its entries are called elements.
- We don't care about order or amount of elements: $\{1, 2, 3, 4\} = \{4, 4, 4, 4, 4, 4, 3, 1, 2\}$.
- $\mathbb{N}$ is the set of natural numbers, $\{1, 2, 3, 4, \ldots\}$. Some people include 0, we will not.
- $\mathbb{Z}$ is the set of integers, $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.
- $\mathbb{Q}$ is the set of rational numbers, or fractions. $\mathbb{R}$ is the set of real numbers, and $\mathbb{C}$ is the set of complex numbers.
- The symbol $\in$ is read "in", and symbolizes belonging. For example, 5 is a natural number, so we could write $5 \in \mathbb{N}$.
- The symbol $\subset$ is read "is a subset of". Notice how all the elements of $\mathbb{N}$ are also in $\mathbb{Z}$. That is, $\mathbb{N}$ is a subset of $\mathbb{Z}$, or $\mathbb{N} \subset \mathbb{Z}$.
- There is also the symbol $\subseteq$. Depending on the author, $\subset$ can be closer to $<$, where one set is in another, but not equal. Then $\subseteq$ is closer to $\leq$, so one set is in the other, and may be equal. Sometimes, to be extra clear, one might write $\subsetneqq$ to show that one set is in the other, and cannot possibly be equal. I will use the symbol $\subset$ to be "less than or equal to", but feel free to use whatever feels most natural to yourself.
- Sometimes, to make things look "pretty", one will also use the symbol $\supset$ or $\ni$.
- There is one set that has no elements, called the empty set, written $\emptyset$. The empty set is in every other set; if A is a set, then $\emptyset \subset A$.

- How to prove a set is a subset of another set? Let A and B be sets. What we do is let x be any arbitrary element of A, that is, $x \in A$. Then we show that x also belongs to B, that is, $x \in B$. It then follows that $A \subset B$.

- How to prove two sets are equal? We show that both $A \subset B$ and $A \supset B$. Then $A = B$. Writing out Venn diagrams can help visualize, but they are not proofs.

- Set builder notation: $\{x \in \text{some set} | \text{x satisfies some requirement}\}$. This is read "the set of x in some set such that x satisfies some requirement". Some people use | and some use : both mean such that.

- For the next points, we will let Ω be the universe. Everything we talk about will live inside Ω, and A and B will be subsets of Ω. We will always assume that the sets we talk about live in some bigger universe.

- Union of two sets: The union of A and B, written $A \cup B$ is the set of everything that is either in A, B, or both. $A \cup B = \{x \in \Omega | x \in A \text{ or } x \in B\}$.

- Intersection of two sets: The intersection of A and B, written $A \cap B$ is the set of everything that is in both A and B. $A \cap B = \{x \in \Omega | x \in A \text{ and } x \in B\}$. If $A \cap B = \emptyset$, we say that A and B are disjoint.

- Difference of two sets: The difference of A and B, written $A \setminus B$ or $A - B$, is the set of everything that is in A but not in B. $A \setminus B = \{x \in A | x \notin B\}$.

- Complement of a set: The complement of A, written $\Omega \setminus A$ or A^{c} (A complement), is the set of everything that is not in A. $A^{\mathsf{c}} = \{x \in \Omega | x \notin A\}$.

- Symmetric difference of two sets: The symmetric difference of A and B, written $A \triangle B$, is the set of everything that is in either A or B, but not both. $A \triangle B = \{x \in \Omega | (x \in A \text{ and } x \notin B) \text{ or } (x \notin A \text{ and } x \in B)\}$.

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Exercises

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For all that follows, we will let Ω be the universe, and A, B, C be subsets. If the question asks "what is", no proof is required.

Exercise 1: What is following set equal to? $\{x \in \mathbb{N} | x^2 = 4\}$
Exercise 2: What is following set equal to? $\{x \in \mathbb{Z} | x^2 = 4\}$
Exercise 3: What is following set equal to? $\{x \in \mathbb{R} | x^2 = -1\}$
Exercise 4: What is following set equal to? $\{x \in \mathbb{C} | x^2 = -1\}$
Exercise 5: Let $\Omega = \mathbb{N}$, $A = \{1, 2, 3, 4\}$, and $B = \{3, 4, 5, 6\}$. What is $A \cup B$, $A \cap B$, $A \setminus B$, $B \setminus A$, and $A \triangle B$?
Exercise 6: Prove that $\emptyset \subset A$, for any set A.
Exercise 7: Let $A = \{\emptyset, \{\emptyset\}\}$. Is $\emptyset \in A$? How about is $\{\emptyset\} \in A$. Is $\{\emptyset\} \subset A$? Lastly, is $\{\{\emptyset\}\} \in A$ or $\{\{\emptyset\}\} \subset A$?
Exercise 8: What is $A \cup \emptyset$ equal to?
Exercise 9: What is $A \cap \emptyset$ equal to?
Exercise 10: What is $\mathbb{Z} - \mathbb{N}$ equal to?
Exercise 11: What do you need to do to prove that $\mathbb{N} \subsetneqq \mathbb{Z}$? How about $\mathbb{R} \subsetneqq \mathbb{C}$. Then prove it.
Exercise 12: Prove $A \cap B \subset B$.
Exercise 13: Prove $A \cap B = B$ if and only if $B \subset A$.
Exercise 14: Prove $A \cup B \supset A$.
Exercise 15: Prove $A \cup B = A$ if and only if $B \subset A$.
Exercise 16: Prove $(A \cup B) \cup C = A \cup (B \cup C)$. Therefore, we just write $A \cup B \cup C$, as the order of operations does not matter.
Exercise 17: Prove $(A \cap B) \cap C = A \cap (B \cap C)$. Therefore, we just write $A \cap B \cap C$, as the order of operations does not matter.
Exercise 18: Prove $(A \cup B)^{\mathsf{c}} = A^{\mathsf{c}} \cap B^{\mathsf{c}}$.
Exercise 19: Prove $(A \cap B)^{\mathsf{c}} = A^{\mathsf{c}} \cup B^{\mathsf{c}}$. This and the previous exercise are called De Morgan's Laws.
Exercise 20: Use De Morgan's law twice to prove $(A \cup B \cup C)^{\mathsf{c}} = A^{\mathsf{c}} \cap B^{\mathsf{c}} \cap C^{\mathsf{c}}$.

Exercise 21: Prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
Exercise 22: Prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. This and the previous exercise are called the distributive laws.
Exercise 23: Prove $(A^{\mathsf{c}})^{\mathsf{c}} = A$.
Exercise 24: Prove $A \setminus B = A \cap B^{\mathsf{c}}$.
Exercise 25: Prove $(A \setminus B)^{\mathsf{c}} = A^{\mathsf{c}} \cup B$.
Exercise 26: What is $A \triangle \emptyset$?
Exercise 27: What is $A \triangle A$?
Exercise 28: What is $\emptyset^{\mathsf{c}}$?
Exercise 29: Prove $A \subset B$ if and only if $B^{\mathsf{c}} \subset A^{\mathsf{c}}$.
Exercise 30: Find sets A and B so that neither $A \subset B$ nor $B \subset A$.
Exercise 31: Prove $A \triangle B = (A \setminus B) \cup (B \setminus A)$.
Exercise 32: Prove $A \triangle B = (A \cup B) \setminus (A \cap B)$.