SPRING 2021 COMPETITION

OLGA RADKO MATH CIRCLE
ADVANCED 3

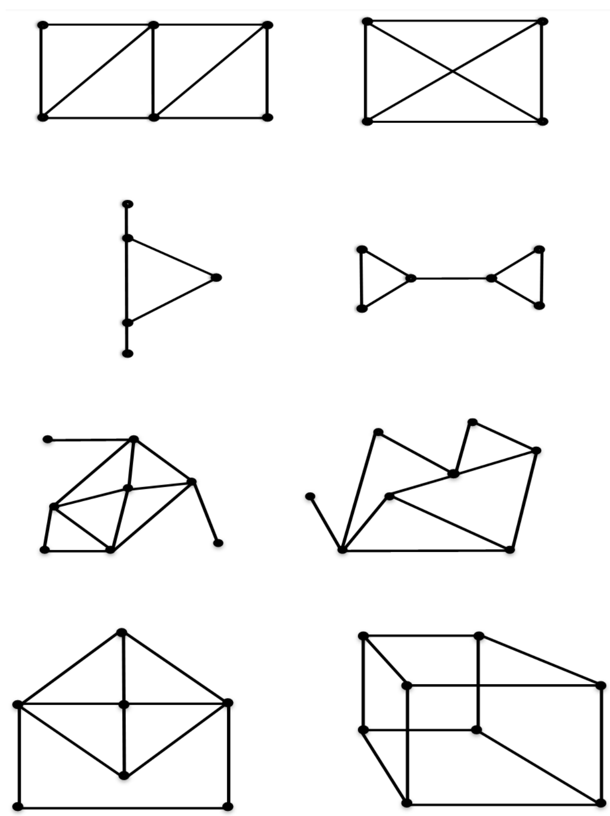
JUNE 6, 2021

Each problem is worth 2 points unless otherwise noted.

1. Graph Theory

Problem 1.1. Prove that any convex polyhedron (not just the platonic solids) must have at least one face with at most 5 sides.

Problem 1.2 (0.5 point per graph, one attempt)**.** Find the chromatic number of each of these graphs. All 8 answers must be submitted together, and you get only one attempt.



Problem 1.3. Let G be a graph such that every subgraph of G contains a vertex of degree at most d. Show that G is $(d+1)$-colorable.

1.1. **Triangle-free Planar Graphs.** Let $G = (V, E)$ be a planar graph containing no triangles (that is, there are no three vertices, all of which are connected to each other).

Problem 1.4. Show that $|E| \leq 2|V| - 4$.

Problem 1.5. Show that G is 4-colorable.

1.2. **Friendship Graphs.** A *friendship graph* is a graph with at least 3 vertices such that every two distinct vertices v, w, there is a unique vertex x that is connected to both v and w. For the following problems, you may use this theorem:

Theorem 1.1. *Any friendship graph contains a vertex that is connected to all other vertices.*

Problem 1.6. Show that if $G = (V, E)$ is a friendship graph, then there is some number n such that $|V| = 2n + 1$. In terms of n, what is $|E|$? Describe the structure of such a graph with a picture.

Problem 1.7. If F_n is a friendship graph with $2n+1$ vertices, calculate the chromatic number and chromatic polynomial of F_n.

1.3. **Ramsey Theory.** Remember that $R(s, t)$, known as a *Ramsey number*, is the smallest number such that any time the edges of $K_{R(s,t)}$ are colored blue and red, there is either a set of s vertices such that all of the edges between them are blue, or a set of t vertices such that all of the edges between them are red.

Problem 1.8. Let G be a graph with vertices $\{0, 1, \ldots, 16\}$ such that m, n are connected by an edge if and only if there is a number k such that $m \equiv n + k^2 \pmod{17}$. Use G to show that $R(4, 4) = 18$.

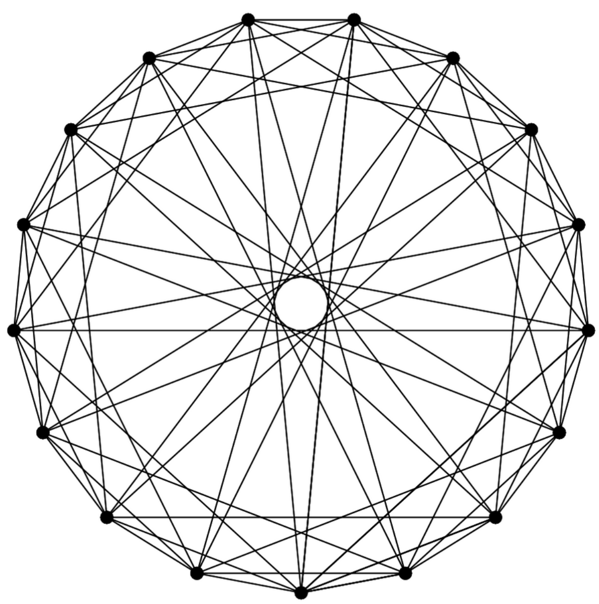


FIGURE 1. G

(Hint: It will be useful to revisit our graph theory worksheet.)

2. Quantum Information

Problem 2.1. Let's play a game! Specifically, the CHSH game. To play this, get two players and ask Aaron. You will go to a separate breakout room, and the game will be played over chat. Aaron will send each of you 0 or 1, which you will keep secret. You will then send back

0 or 1. If you both got sent 1, then you win if you send back the same number, otherwise, you win if you send back different numbers.

You can play up to 4 times, each time you win, you get 0.5 points.

Problem 2.2. Now let's play Quantum CHSH! The same setup as regular CHSH, except you can message me to measure entangled qubits, which I will faithfully simulate.

You can play up to 4 times, each time you win, you get 0.5 points.

3. Number Theory

Problem 3.1. Prove that if p is a prime number, then $(p-1)! \equiv -1 \pmod{p}$.

Problem 3.2. Prove that if p is a prime, there are integers x, y such that $x^2 + y^2 + 1$ is divisible by p.

Problem 3.3. Find a constant C such that for any k, $\log\left(1 + \frac{1}{n} + \cdots + \frac{1}{n^k}\right) \leq \frac{C}{n}$.

Problem 3.4. Let $p_1, p_2, p_3, \ldots$ be the primes listed in increasing order. Show that there is a constant C such that the sum $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} \geq C \log(\log(p_k))$. This gives another proof that $\sum_{i=1}^{\infty} \frac{1}{p_i}$ diverges. (Hint: You may use the previous problem.)

3.1. **Convolution.** As we've seen in the past, $\tau(n)$ and $\sigma(n)$ are examples of *arithmetic functions*: their domain is the set of (positive) natural numbers $\mathbb{N}$.

Definition 3.1. Given two arithmetic functions f, g, we define their *convolution* as

$$f * g(n) = \sum_{d|n} f(d)g(n/d)$$

Problem 3.5.

- Let $u(n) = 1$ for all n. What is $u * u(n)$?
- Let $N(n) = n$. What is $N * u(n)$?

Problem 3.6. What function $e(n)$ has the property that $f * e(n) = f(n)$ for all arithmetic functions f and positive integers n?

Problem 3.7. What function $\mu(n)$ has the property that $\mu * u(n) = e(n)$ for all positive integers n? (Hint: calculate it on prime powers first.)

4. Other topics

Problem 4.1. The Mayor of Los Angeles is planning to split the city into 27 districts, such that each district shares a border with exactly 5 other districts. A cartographer tried and failed to draw a map that fulfills the Mayor's request. Can you help the cartographer and show that such a map cannot exist?

Problem 4.2. Gather a team of 4 players and find Alvin for a hat puzzle. Each member of your team is to be fitted with a red or blue hat. Each player will be told the colors of the hat of their teammates, but not the color of their own hat. No communications are permitted during the game. At a signal, each player will tell the color of their own hat through private chat to the instructors. You win the game if at least two players guess their hat's color correctly.

We will play two rounds of this game with every team, and the team gains 1 points for each successful round.

Problem 4.3. Show that when expanded, the product

$$(1 + x + x^2 + x^3 + \cdots + x^{100})(1 - x + x^2 - x^3 + \cdots + x^{100})$$

has no terms of odd degree.

Problem 4.4. Compute the product

$$\cos\frac{\pi}{4} \times \cos\frac{\pi}{8} \times \cos\frac{\pi}{16} \times \ldots \cos\frac{\pi}{2^n} \times \ldots$$

Problem 4.5. Given n red points and n blue points on the plane, no three on a line, prove that there is a matching between them so that line segments from each red point to its corresponding blue point do not cross.

Problem 4.6. Show that every integer can be written as a sum of five cubes of (not necessarily positive) integers.

Problem 4.7. Let $m \geq 3$ and $n \geq 3$ be integers. Let $A := a_{i,j}$ ($i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$) be an $m \times n$ matrix such that every entry in the first row and in the first column is equal to 0. Suppose that the entries of the matrix also satisfies the linear equations of the form

$$a_{i,j} \;=\; \frac{a_{i-1,j} + a_{i+1,j} + a_{i,j-1} + a_{i,j+1}}{4} \qquad 1 < i < m,\ 1 < j < n.$$

How many matrices (for fixed m and n) satisfy these properties?

Problem 4.8. The names of one hundred prisoners are placed into one hundred boxes, one name to a box. The boxes are lined up on a long table in a secret room. Each prisoner is led into the secret room separately, and each may look in at most fifty boxes. They must leave the room exactly the same as before they enter, and they are not allow to communicate with other prisoners after the visit.

After all prisoners finish their visit, every single one of the prisoners need to have found their own name, or all of them will be executed. The prisoners are given time to discuss their strategy in advance.

Find a strategy for them which which has probability of success exceeding 30%.