THIS IS FUN! SPRING EDITION

OLGA RADKO MATH CIRCLE
ADVANCED 2
JUNE 6TH, 2021

1. Measurement Errors

Problem 1. (3 points each sub-problem, and 1 additional point for solving all)

(1) Let $y = e^x$. Use the fact that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ to find $\Delta(y)$ in terms of x and $\Delta(x)$. Make sure to simplify your answer as much as possible.
(2) Now suppose $z = e^{ax}$. Find $\Delta(z)$ in terms of x and $\Delta(x)$. It may help to first find it in terms of ax and $\Delta(ax)$.
(3) The spread of a virus can often be modeled by exponential growth (at least at the very beginning). The number of infected people on day t, $y(t)$, is represented by the equation $y(t) = e^{at}$, where $a = \frac{\ln 2}{\tau}$ and τ is the time it takes for the number of infected people to double. Suppose the number of infected people doubles every 6 days. Find the error $\Delta(y)$ on day 18.

Problem 2. (10 points) You have a spherical non-water balloon full of water. Unfortunately, you're not allowed to keep it. Instead, you must empty it into a cubic box and go to bed, *without dinner.* I forgot to mention, you have so many cubes of all sizes. Being the space-considerate person that you are, you want to choose the smallest cube that will hold **all** the water from the non-water balloon. You measure the diameter of the non-water balloon to be 11 inches. Use the measurement error of the volume to find the side length of the cube. (`Make sure that no water is spilled`.)

Problem 3. (10 points) You and your buddy Buddy are so excited to approximate π. His son Sonny comes up with the best idea some side of the Mississippi. You will construct a perfect circle. Nature does it all the time, right? How hard can it be? Buddy will measure the diameter of the circle. Sonny's pal Paul will take a string and wrap it around the circle to measure the circumference. Being a good Math Circle student, you know how to get π given the diameter and the circumference of a circle. Buddy measures the diameter to be 45 feet and Paul measures the circumference to be 138 feet. Use the concept of measurement errors to explain why at least one of Sonny's dad and friend is terrible at measuring things.

Problem 4. (5 points) The Mars Climate Orbiter (formerly the Mars Surveyor '98 Orbiter) was a 638-kilogram (1,407 lb) robotic space probe launched by NASA on December 11, 1998 to study the Martian climate, Martian atmosphere, and surface changes and to act as the communications relay in the Mars Surveyor '98 program for Mars Polar Lander. However, on September 23, 1999, communication with the spacecraft was permanently lost as it went into orbital insertion. The spacecraft encountered Mars on a trajectory that brought it too close to the planet, and it was either destroyed in the atmosphere or escaped the planet's vicinity and entered an orbit around the Sun. An investigation attributed the failure to a measurement mismatch between two software systems: metric units by NASA and non-metric (imperial or "English") units by spacecraft builder Lockheed Martin. Now that's what I call a *measurement error*!

Rate this joke from 0 to 5, 0 being zero and 5 being five.

2. Smart Codes

Problem 5. (5 points + 1 bonus point) The instructor will give you a sequence of 10 digits. Check if it is a valid ISBN number. If not, change the check bit so that it is valid. Figure out to which book this ISBN refers. If you can help the instructor decide whether it is worth reading, you will receive one bonus point.

Problem 6. (5 points) What is the information rate of repeating $[n, k]$-code in terms of k and n?

Problem 7. (5 points) Can repeating $[4, 2]$-code detect single digit error? You will receive points only with a good explanation.

Problem 8. (1 point) What's your favorite book?

Problem 9. (5 points each) Determine which of the following are valid Hamming's square codewords. If not valid, figure out whether there is transposition or single bit error. Fix the error.

(a) 110000101
(b) 110010110
(c) 110011101

Problem 10. (10 points or -5 points based on the result of a post problem coin flip) The instructor will give you a decimal integer that has a four bit binary expansion. You have 2 minutes to convert the number to binary and encode the four bits using Hamming's square code.

Problem 11. (The point value is the random number generated by Orbo) Ditto Problem 10 but Orbo, a random number generator, produces the integer. Race the instructor. If you win, your team gets the points. If the instructor wins, they kick you out of the meeting.

Problem 12. (5 points) What is the information rate of Hamming's 6×4 rectangular code?

Halfway done. Remember to take a snack break.

3. Sequences

Problem 13. (10 points) A hiker begins an ascent of Mount Whitney on Saturday morning, reaching the summit by nightfall. She then spends the night at the summit, and starts down the mountain the following morning, reaching the bottom on Sunday nightfall. Prove that, at some precise time of day, they was exactly at the same altitude on Tuesday as he was on Monday.

Problem 14. (5 points) Determine if $d(x, y) = |x^2 - y^2|$ is a metric on $\mathbb{R}$. If not, which property of a metric is not satisfied?

Problem 15. (10 points) Give an example of a continuous function for which the Intermediate Value Theorem does not hold. (Hint: Pay attention to the domain of the function.)

Problem 16. (10 points) Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x^3 - x + 100$. Find an interval of length at most 1 on which f has a zero.

Problem 17. (5 points) A man was born on the island Crete off the coast of Greece. One day while shopping for Terra Chips$^{\mathrm{TM}}$ at the "local" supermarket, he introduces himself to the shop owner whose name happens to be Richard. What do you call this situation?

4. Miscellaneous

Problem 18. (5 points for each round) Gather a team of 4 players and find an instructor for a hat puzzle (an instructor will join your team if you don't have enough players, and you need to explain your strategy to the instructor). Each member of your team is to be fitted with a red or blue hat. Each player will be told the colors of the hat of their teammates, but not the color of their own hat. No communications are permitted during the game. At a signal, each player will tell the color of their own hat through private chat to the instructors. You win the game if at least two players guess their hat's color correctly.

We will play two rounds of this game with every team, and the team gains 5 points for each successful round.

Problem 19. (10 points) Lockers numbered 1 to 100 stand in a row at a school building. When the first student arrives, she opens all the lockers. The second student then goes through and recloses all the even-numbered lockers. The third student changes every locker whose number is a multiple of 3. This continues until 100 students have passed through. Which lockers are now open?

Problem 20. (10 points) Are there any integer solutions to the equation $x^2 + 3y^2 = 360$? Prove it.

Problem 21. (10 points) You have an opportunity to bet 1 dollar on a number between 1 and 6. Three dice are then rolled. If your number fails to appear, you lose your 1 dollar. If it appears once, you win 1 dollar; if twice, 2 dollars; if three times, 3 dollars.

Is this game in your favor, fair, or against the odds?

Problem 22. (10 points) Show that every integer divisible by 6 can be written as a sum of four cubes of (not necessarily positive) integers.

Problem 23. (10 points+1 bonus point) A calculator is programmed to calculate a fixed polynomial P with positive integer coefficients and unknown degree. When you input a positive integer n, it will display the value $P(n)$, which counts as one step. You are allowed to repeat this process. Try to figure out the polynomial P in as few steps as possible.

When you submit your answer, the instructor will have two polynomials prepared in advance. You will be asked to guess the polynomial after 6 steps (for practical concerns), and if correct you will get at most 5 points depending on how many steps are taken (the fewer, the better), for a total of 10 points maximum. You cannot retry this problem.

Bonus: what if you are allowed to evaluate at any real value? Can you do it in one step?

Problem 24. (10 points) Given n red points and n blue points on the plane, no three on a line, prove that there is a matching between them so that line segments from each red point to its corresponding blue point do not cross.

Problem 25. (20 points) Given a connected graph, a **Hamiltonian path** from v to w is a path which starts at v, ends at w, and visits every vertex exactly once.
Consider the continental United States (i.e., without Hawaii and Alaska) and view it as a graph in the following manner:

(1) The vertices are the states
(2) Two vertices share an edge if the states share a border

Notice that Maine only borders one state, so any Hamiltonian path would have to either start or end in Maine. Suppose we start in Maine. New York is what's called a *bottleneck*; removing New York would disconnect the graph. Therefore, any Hamiltonian path which starts at Maine would have to end on the other side of New York (which we will call the Western US). An all knowing computer has verified that there are Hamiltonian paths ending at every single state west of New York **except one**.

Which state is the exception?