TAXICAB GEOMETRY

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Euclidean Geometry

In geometry the primary objects of study are *points*, *lines*, *angles*, and *distances*. We can identify each point in the plane by its Cartesian coordinates (x, y). The *Euclidean distance* between points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ is

$$d_E(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \qquad \text{(Euclidean distance formula)}$$

Taxicab Geometry

In "taxicab" geometry, *the points, lines, and angles are the same,* but the *notion of distance is different* from the Euclidean distance. The *taxicab distance* between points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ is

$$d_T(A, B) = |x_2 - x_1| + |y_2 - y_1|. \qquad \text{(Taxicab distance formula)}$$

Exercises.

(1) On a sheet of graph paper, mark each pair of points P and Q and find the both the Euclidean and taxicab distance between them:
  (a) $P = (0, 0)$, $Q = (1, 1)$ (b) $P = (1, 2)$, $Q = (2, 3)$ (c) $P = (1, 0)$, $Q = (5, 0)$

(2) (Taxi circles) Let A be the point with coordinates $(2, 2)$.
  (a) Plot A on a piece of graph paper, and mark all points P such that $d_T(A, P) = 1$. The set of such points is written $\{P \mid d_T(A, P) = 1\}$. Also mark all points in $\{P \mid d_T(A, P) = 2\}$.
  (b) Graph the set of points which are a distance 2 from the point $B = (1, 1)$.

(3) (Lines) Recall that in taxicab geometry, the shapes we call lines are the same as the usual lines in Euclidean geometry. Graph the line ℓ that passes through the points $(3, 0)$ and $(0, 3)$.
  (a) If A is the point $(4, 2)$ what is the Euclidean distance to ℓ? (What is meant by the distance from a point to a line?)
  (b) What is a reasonable notion for the taxicab distance from the point A to the line ℓ? How does it compare with the Euclidean notion?

(4) If $P = (0, 0)$ and $Q = (3, 4)$, mark the set of points A such that $d_T(A, P) = d_T(A, Q)$. What shape is the result?

(5) Let $A = (0, 0)$ and $B = (5, 3)$.
  (a) Find $d_T(A, B)$.
  (b) Graph $\{P \mid d_T(A, P) = 4, d_T(B, P) = 4\}$.
  (c) Graph $\{P \mid d_T(A, P) = 5, d_T(B, P) = 3\}$.
  (d) Graph $\{P \mid d_T(A, P) = 1, d_T(B, P) = 7\}$.
  (e) Graph $\{P \mid d_T(A, P) + d_T(B, P) = d_T(A, B)\}$.
  (f) Repeat the previous step with d_E instead of d_T. What is the resulting shape?

Material and problems adapted from Eugene F. Krause, *Taxicab Geometry*, Dover 1986.

(6) Let $A = (1,1)$ and $B = (4,6)$. Mark each of the following sets:
   (a) $\{P \mid d_T(P,A) + d_T(P,B) = 10\}$
   (b) $\{P \mid d_T(P,A) + d_T(P,B) = 8\}$
   (c) $\{P \mid d_T(P,A) + d_T(P,B) \leq 2\}$
(7) Sketch $\{P \mid d_T(P,\ell) = d_T(P,F)\}$ and $\{P \mid d_E(P,\ell) = d_E(P,F)\}$ where:
   (a) ℓ is the line passing through $(0,0)$ and $(2,-1)$, and $F = (1,1)$.
   (b) ℓ is the line passing through $(-2,2)$ and $(2,0)$, and $F = (5,5)$.
   If we sketch a parabola, can we always assume the focus lies on the y-axis? Can we assume the directrix is horizontal?
(8) Sketch $\{P \mid d_T(P,A) - d_T(P,B)| = 1\}$ where:
   (a) $A = (-2,0$ and $B = (2,0)$.
   (b) $A = (0,0)$ and $B = (2,2)$.
   (c) $A = (0,0)$ and $B = (3,4)$.
   If we sketch a hyperbola, can we always assume one of the vertices is at the origin? Can we assume they both lie on the x-axis?
(9) Mark the points (-3,0), (-5,4), and (3,0).
   (a) Sketch all points in the *interior* of $\angle BAC$ which are the same taxi distance from the sides of the angle. (What is meant by the interior of an angle?)
   (b) Repeat the exercise with Euclidean distance.
(10) Using the same points as in the previous problem
   (a) Circumscribe a Euclidean circle around the triangle.
   (b) Repeat the exercise, circumscribing a taxicab circle.