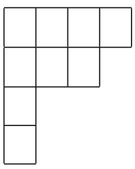
Lesson 6: Young Diagrams

Konstantin Miagkov

Definition 1.
The *partition number* $p(n)$ for a positive integer n is the number of partitions of n into positive parts where partitions different only in ordering of the summands are not distinguished. So for example $9 = 1 + 3 + 1 + 4$ and $9 = 4 + 3 + 1 + 1$ correspond to the same partition.

Definition 2.
The *Young diagram* is a finite collection of boxes, or cells, arranged in left-justified rows, with the row lengths in non-increasing order.
Listing the number of boxes in each row gives a partition of a non-negative integer n, the total number of boxes of the diagram. That gives a one-to-one correspondence between partitions and Young diagrams.
For example the following diagram corresponds to the partition $9 = 4 + 3 + 1 + 1$:



Problem 1.
Compute $p(n)$ for all n from 1 to 8.

Problem 2.
You need to pack cookies into boxes. There are 10 boxes each of which can contain at most 3 cookies. How many ways are there to put 22 cookies into boxes (leaving no box empty)? The boxes are indistinguishable.

Problem 3.
Show that the number of partitions of n into at most k parts each of which is at most ℓ is equal to the number of partitions of n into at most ℓ parts each of which is at most k.

Problem 4.
Show that the number of partitions of n into k parts is equal to the number of partitions of $n + \binom{k}{2}$ into k *distinct* parts.

Problem 5.
Show that the number of partitions of n into distinct odd parts is equal to the number of partitions of n such that their Young diagrams are symmetric with respect to the diagonal.

Problem 6.

Let the side lengths of triangle $\triangle ABC$ be a, b, c where a is the length of BC, b is the length of AC and c is the length of AB. Let M, N be points on AB and BC respectively such that $AM = BN$ and MN is parallel to AC. Find the length of MN in terms of a, b, c.

Problem 7.

Consider points A, B, C, D on a line l in that order. Draw two parallel lines through points A and B, and another pair of parallel lines through points C and D. The two pairs of parallel lines create a parallelogram. Consider the two points at which the lines containing the diagonals of this parallelogram intersect l. Show that these two points do not depend on the choice of the two pairs of parallel lines.