Lesson 6: Young Diagrams

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**Definition 1.**
The partition number $p(n)$ for a positive integer $n$ is the number of partitions of $n$ into positive parts where partitions different only in ordering of the summands are not distinguished. So for example $9 = 1 + 3 + 1 + 4$ and $9 = 4 + 3 + 1 + 1$ correspond to the same partition.

**Definition 2.**
The Young diagram is a finite collection of boxes, or cells, arranged in left-justified rows, with the row lengths in non-increasing order.
Listing the number of boxes in each row gives a partition of a non-negative integer $n$, the total number of boxes of the diagram. That gives a one-to-one correspondence between partitions and Young diagrams.
For example the following diagram corresponds to the partition $9 = 4 + 3 + 1 + 1$:

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**Problem 1.**
Compute $p(n)$ for all $n$ from 1 to 8.

**Problem 2.**
You need to pack cookies into boxes. There are 10 boxes each of which can contain at most 3 cookies. How many ways are there to put 22 cookies into boxes (leaving no box empty)? The boxes are indistinguishable.

**Problem 3.**
Show that the number of partitions of $n$ into at most $k$ parts each of which is at most $\ell$ is equal to the number of partitions of $n$ into at most $\ell$ parts each of which is at most $k$.

**Problem 4.**
Show that the number of partitions of $n$ into $k$ parts is equal to the number of partitions of $n + \binom{k}{2}$ into $k$ distinct parts.

**Problem 5.**
Show that the number of partitions of $n$ into distinct odd parts is equal to the number of partitions of $n$ such that their Young diagrams are symmetric with respect to the diagonal.
Problem 6.
Let the side lengths of triangle $\triangle ABC$ be $a, b, c$ where $a$ is the length of $BC$, $b$ is the length of $AC$ and $c$ is the length of $AB$. Let $M, N$ be points on $AB$ and $BC$ respectively such that $AM = BN$ and $MN$ is parallel to $AC$. Find the length of $MN$ in terms of $a, b, c$.

Problem 7.
Consider points $A, B, C, D$ on a line $l$ in that order. Draw two parallel lines through points $A$ and $B$, and another pair of parallel lines through points $C$ and $D$. The two pairs of parallel lines create a parallelogram. Consider the two points at which the lines containing the diagonals of this parallelogram intersect $l$. Show that these two points do not depend on the choice of the two pairs of parallel lines.