ORMC: LINEAR TRANSFORMATIONS

OLYMPIAD GROUP 1, WEEK 7

Today we will focus on a quite general class of transformations of Euclidean space, called *linear transformations*; we've actually seen many examples of these before. If we think of points in $\mathbb{R}^2$ as vectors $\vec{v} = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$, a *linear* transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is any map with the properties

$$T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w}) \qquad \text{and} \qquad T(\alpha\vec{v}) = \alpha\vec{v},$$

for any real α. Here, the additions of vectors and their scalings by α have the obvious meanings:

$$\begin{pmatrix} 2 \\ 5 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}, \qquad 3\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 6 \\ 15 \end{pmatrix}.$$

Linear transformations are those which *preserve* these "linear" operations of addition and scaling (the same definition generalizes to $\mathbb{R}^3$, $\mathbb{R}^4$, etc.). Equivalently, they preserve linear combinations:

$$T(\alpha\vec{v} + \beta\vec{w}) = \alpha T(\vec{v}) + \beta T(\vec{w}).$$

Now we turn to a way to represent such transformations compactly. The key observation is that any vector $\vec{v} = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ can be represented as a linear combination of the *standard basis vectors* $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$:

$$\begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

It follows that for any linear transformation T,

$$T\begin{pmatrix} x \\ y \end{pmatrix} = xT\begin{pmatrix} 1 \\ 0 \end{pmatrix} + yT\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

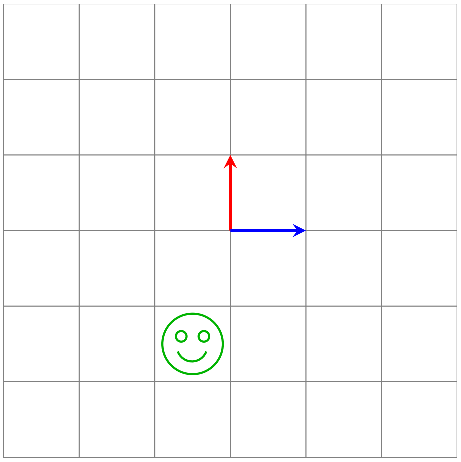
so to describe T, it suffices to know the values of $T\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $T\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. We can thus represent T as a 2×2 matrix:

$$T \quad \longleftrightarrow \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \text{where } T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \text{ and } T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}.$$

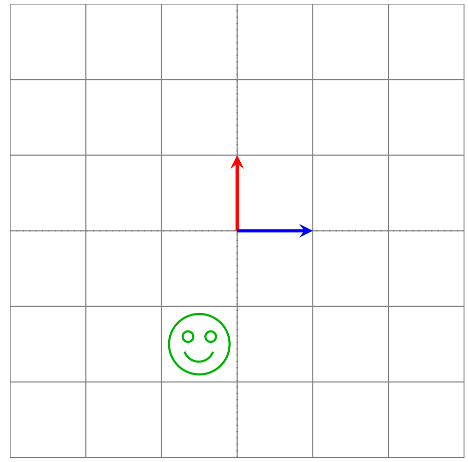
The general resulting formula is

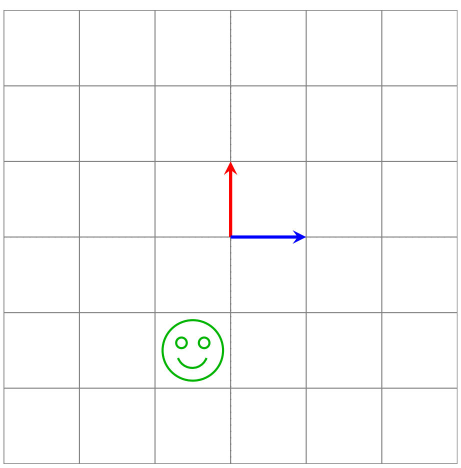
$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} a \\ c \end{pmatrix} + y\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.}$$

Below are a few pictures of how some linear transformations (represented as 2×2 matrices) transform the Euclidean plane. On the left, the blue and red vectors are $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$.

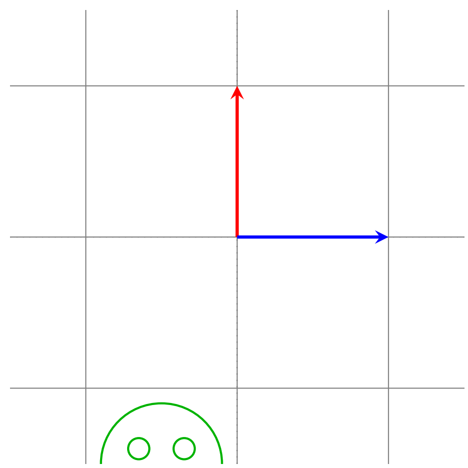


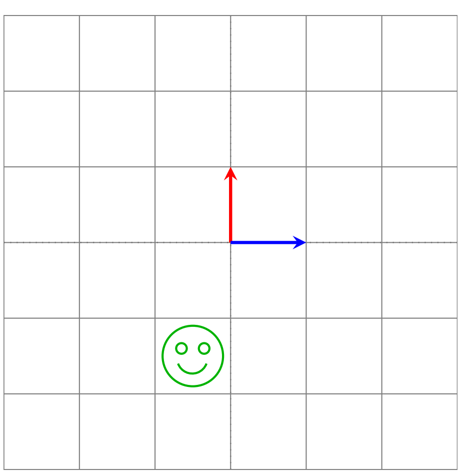
$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \longrightarrow$$



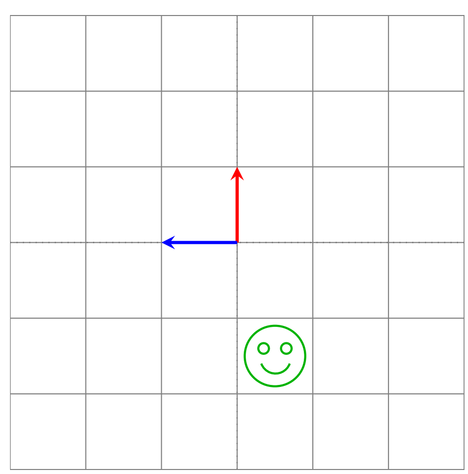


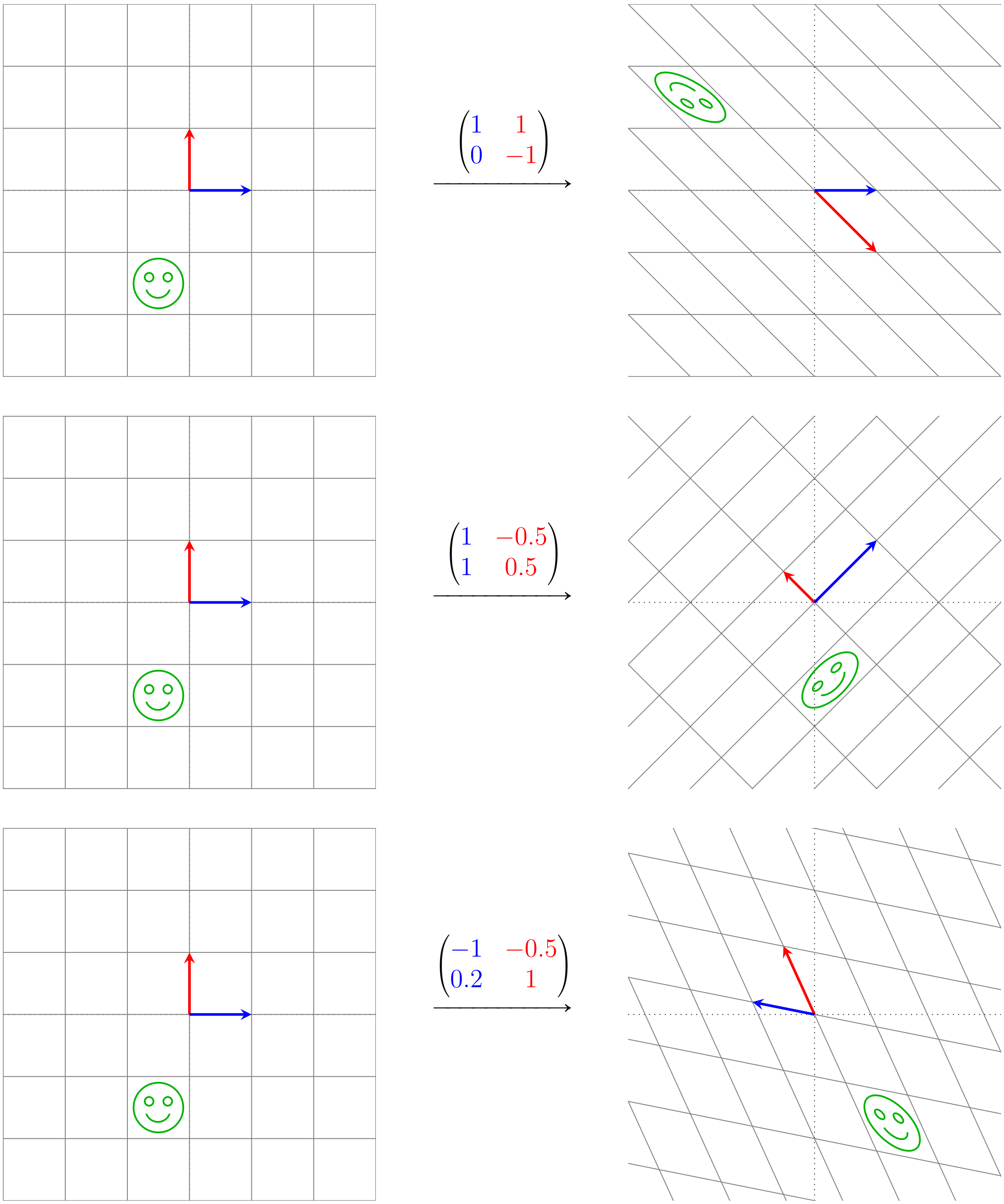
$$\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix} \longrightarrow$$





$$\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix} \longrightarrow$$





By looking at what happens to a vector $\binom{x}{y}$ (or, without loss of generality, to the standard basis vectors), it makes sense to define the addition and multiplication of two matrices:

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} m & n \\ p & q \end{pmatrix} = \begin{pmatrix} a+m & b+n \\ c+p & d+q \end{pmatrix}}$$

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} m & n \\ p & q \end{pmatrix} = \begin{pmatrix} am+bp & an+bq \\ cm+dp & cn+dq \end{pmatrix}.}$$

These correspond to what happens when we add, respectively *compose*, linear transformations. Indeed, if A, B are 2×2 matrices and $\vec{v} \in \mathbb{R}^2$ is a vector, we have

$$(A+B)\vec{v} = A\vec{v} + B\vec{v}, \qquad \text{and} \qquad (AB)\vec{v} = A(B\vec{v}).$$

WARNING: The multiplication of matrices is not generally commutative! For example,

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Problem 1. (Homotheties)

(a) Show that the *identity matrix* $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ satisfies

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

for any $x, y, a, b, c, d \in \mathbb{R}$.

(b) Let $\lambda \in \mathbb{R}$. Show that the matrix $\lambda I = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right)$ satisfies

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

for any $x, y, a, b, c, d \in \mathbb{R}$. Does this relate to any type of transformation that you've seen before?

Problem 2. (Rotational homotheties and reflections)

(a) Let a, b be a real numbers. If we associate to the complex number $z = x + yi$ the vector $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$, show that the complex map

$$z \mapsto (a + bi) \cdot z$$

corresponds to the linear transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Remark. You can thus think of 2×2 matrices as a *generalization* of complex numbers.

(b) What does a rotation by α radians (counterclockwise) with respect to the origin look like, as a 2×2 matrix?

(c) What does the reflection with respect to the real axis, $z \mapsto \overline{z}$, look like as a matrix?

Problem 3. (Determinants and areas) Define the *determinant* of a 2×2 matrix by

$$\boxed{\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} := ad - bc.}$$

(a) Show that the image of the unit square $\{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) : 0 \leqslant x, y \leqslant 1\}$ under the linear transformation $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a parallelogram of area

$$\left| \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right|.$$

Remark: This holds for more general shapes too, so you can think of the determinant as a homothety factor for areas.

(b) What happens when the determinant is zero?

(c*) Show that for any 2×2 matrices A, B, one has

$$\det(AB) = \det(A) \cdot \det(B).$$

Explain why this makes sense in terms of *homothety factors for areas.*

(d) What's the determinant of the linear transformation $z \mapsto (a+bi) \cdot z$ from Problem 2(a)? What's the analogue of part (c) above in terms of complex numbers?

Problem 4. (Inverses and systems of linear equations)

(a) Fix real constants a, b, c, d. Try to solve for x and y in the system

$$\begin{cases} ax + by = 2 \\ cx + dy = 5. \end{cases}$$

What needs to happen so that there is exactly one solution $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$? How does this relate to Problem 3(b)?

(b) Denote $A := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, and assume that $\det(A) \neq 0$. Consider the matrix

$$A^{-1} := \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Show that $A^{-1} \cdot A = A \cdot A^{-1} = I$.

(c) Verify that the linear system in part (a) can be represented as the matrix-vector equation

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix},$$

where we denoted $A := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Left-multiply both sides by A^{-1} to solve the linear system.

(d) Suppose that $A = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right)$ is a homothety by λ, and $B = \left(\begin{smallmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{smallmatrix}\right)$ is a rotation by α radians. What do A^{-1} and B^{-1} look like?

Problem 5. (Fibonacci numbers)

(a) Show that the Fibonacci numbers (given by $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 2$), satisfy

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix}.$$

(b) Conclude that for all $n \geqslant 1$,

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(c) You can assume that

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = A \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix} A^{-1},$$

where $A = \left(\begin{smallmatrix} 1+\sqrt{5} & 1-\sqrt{5} \\ 2 & 2 \end{smallmatrix}\right)$. Show that

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = A \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1+\sqrt{5}}{2}\right)^n \end{pmatrix} A^{-1}.$$

(d*) Deduce that for all $n \geqslant 0$,

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right).$$

[3×3 *matrices: linear transformations on* $\mathbb{R}^3$*; determinants are homothety factors for volumes*]

Problem 6. (Rotations in 3D)

(a) What does the identity 3×3 matrix look like? How about a homothety by λ?

(b) Show that the matrices

$$\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix}$$

correspond to rotations in $\mathbb{R}^3$ around the z-axis, the x-axis, and the y-axis respectively.

(c) Set $\alpha = \pi/2 = 90^\circ$, and verify that no two of the corresponding 3D-rotations commute.

(d) Define the determinant of a 3×3 matrix by

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix} := aej + bfg + cdh - ceg - afh - bdj.$$

($\searrow$diagonals minus $\swarrow$diagonals)

Compute the determinants of the matrices in part (b).

Homework

Problem 1. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix. Show that $\det A = 0$ if and only if there exists a *nonzero* vector $v = \begin{pmatrix} x \\ y \end{pmatrix}$ satisfying $Av = 0$. *Note: this is in fact true for a square matrix of arbitrary size, not just 2×2; however, it is not so easy to prove.*

Problem 2. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix, and let $v_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, v_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ be two nonzero vectors such that $Av_1 = 7v_1$ and $Av_2 = 13v_2$. If we let $V = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$, show that

$$AV = \begin{pmatrix} 7 & 0 \\ 0 & 13 \end{pmatrix} V.$$

If we further assume that V is invertible, find a formula for A^n. *Hint: Recall what we did for the Fibonacci numbers.*