AXIOMS OF GEOMETRY

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While Euclid's *Elements* is considered the classic geometry text, his axioms are not entirely rigorous. Here are Euclid's axioms, roughly translated:

- "To draw a straight line from any point to any other point."
- "To produce a finite straight line continuously in a straight line"
- "To describe a circle withany center and distance"
- "That all right angles are equal to one another"
- (The Parallel Property) "That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles."

Also included are various "common notions" and definitions. However, he defines a point as "that which has no part". While this evokes an intuitive picture, it does not add solid information. In the late 19th century David Hilbert, among others, produced more extensive systems of axioms formalizing the parts that Euclid brushed over. Here is one version (Hilbert's original construction had 21 axioms):

- Given any two points, there is exactly one line containing them.
- Any line contains at least two points. There exist at least three non-collinear points.
- (Distance) To each ordered pair of points P, Q, there is a distance $d(P, Q) \geq 0$ which satisfies the following properties:
  - (Positive definiteness) $d(P, Q) = 0$ if and only if $P = Q$.
  - (Symmetry) For any points P, Q, we have $d(P, Q) = d(Q, P)$.
  - (Triangle inequality) For any points P, Q, R, we have $d(P, R) \leq d(P, Q) + d(Q, R)$.
- For any line ℓ, there is a one-to-one function f_ℓ mapping the points of ℓ to the set of real numbers, $\mathbf{R}$, such that $|f_\ell(P) - f_\ell(Q)| = d(P, Q)$.
- (Plane Separation) Given a line ℓ there are subsets H_1, H_2 ("half-planes") such that
  - H_1 and H_2 are *convex sets* (i.e. if A, B lie in one of them, then so do all points in $\overline{AB}$)
  - $H_1 \cup H_2 = \mathcal{P} \setminus \ell$
  - If $A \in H_1$ and $B \in H_2$ then $\overline{AB} \cap \ell \neq \varnothing$
- (Angle Properties)
  - An angle has a measure between 0 and 180 degrees.
  - Given a ray $\overrightarrow{AB}$ on a half-plane H and a real number r between 0 and 180 there is exactly one ray $\overrightarrow{AP}$ such that $m\angle PAB = r$.
  - If P is in the interior of $\angle ABC$, then $m\angle ABP + m\angle PBC = m\angle ABC$.
  - If B is between A and C and $D \notin \overleftrightarrow{AC}$, then $m\angle ABD + m\angle DBC = 180$.
- (Side-Angle-Side Property) Given a bijective correspondence between the vertices of two triangles, if two sides and the included angle of one triangle are congruent to two sides and the included angle of the other triangle, then the correspondence is a congruence.
- (Parallel Property) Given a line ℓ and a point P not contained in the line ℓ, there is precisely one line passing through P which is parallel to ℓ.

Questions:

- What concepts are necessary to define to state the above axioms rigorously?
- Which of the axioms are satisfied in taxicab geometry? To start, you should decide which concepts are the same and different.