Sequences II: Continuous Functions and the Intermediate Value Theorem

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May 23, 2021

1 Continuous Functions

We begin with an intuitive definition which may be familiar from algebra class.

Definition 1 *$f : \mathbb{R} \to \mathbb{R}$ is **continuous** if its graph can be drawn without lifting the pen.*

Problem 1 *Decide if the following functions $f : \mathbb{R} \to \mathbb{R}$ are continuous.*

a. $f(x) = x^3$

b. $f(x) = \sin(x)$

c. $f(x) = \begin{cases} -1 & x \leq 0 \\ 1 & x > 0 \end{cases}$

d. $f(x) = e^x$

As it turns out, this intuitive definition does not quite suffice for certain applications. In order to refine our notion of what it means for a function to be continuous, we must look back to our study of metric spaces.

Definition 2 *Let $(X, d_X), (Y, d_Y)$ be metric spaces and $x \in X$ be a point. A function $f : X \to Y$ is **continuous at** x if whenever $x_n \to x$ in X, $f(x_n) \to f(x)$ in $Y^{\dagger}$.*

*f is **continuous** if it is continuous at every $x \in X$.*

While continuity varies depending on which metric we use (as we shall see later), the most important application will have to do with $\mathbb{R}$ using the standard metric $d(x, y) = |x - y|$. For now, we will restrict our attention to the case when both the domain and the range have the standard metric.

Problem 2 *Consider the function f as in Problem 1c.*

a. Consider the sequence $(x_n)_{n=1}^{\infty}$ where $x_n = 1/n$. Show that $x_n \to 0$. (Recall last week!)

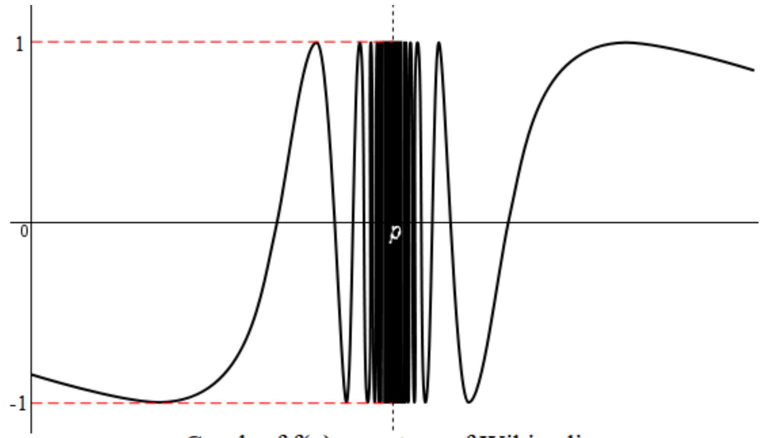
b. Show that $f(x_n) \to 1$.

c. Explain why f is not continuous at 0.

The reason we needed Definition 2 is because there are certain functions which can be graphed without lifting the pen, but are not continuous. For instance, consider the function

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

† Once again our definition of continuity appears different from what one might see in, say a calculus textbook. Interested students are encouraged to read about why these two definitions are in fact equivalent.



Graph of f(x), courtesy of Wikipedia

It turns out that f is *not* continuous.

Problem 3 *(Challenge) Prove that $f(x)$ is not continuous at $x = 0$.*

On the other hand, every continuous function certainly does have the property that its graph can be drawn without lifting the pen. This property is encapsulated in the following theorem, which may be familiar from algebra class.

2 The Intermediate Value Theorem

Theorem 1 *(Intermediate Value Theorem) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose there are points $a \neq b$ where $f(a) < 0$ and $f(b) > 0$. Then there is a point c between a and b such that $f(c) = 0$.*

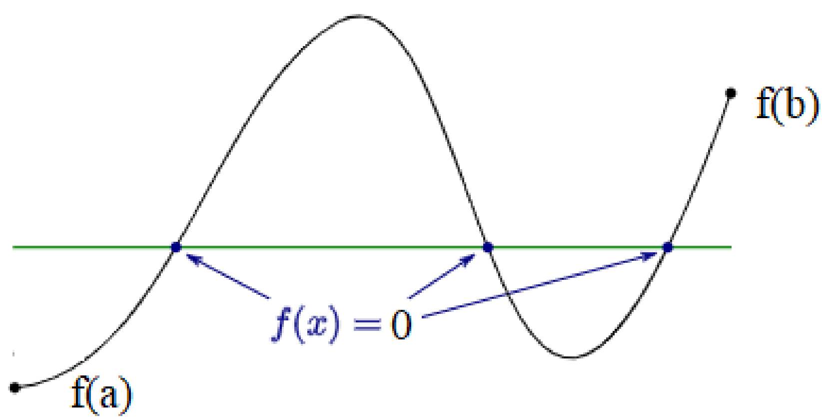


Image courtesy of Wikipedia

We will not prove the Intermediate Value Theorem, but we will consider many of its applications. As an example of how to apply the IVT, consider the height of a dolphin as it swims in the ocean. We can write its height as a function $h(t)$ of time, and unless this dolphin can teleport, h will be continuous. In this example, the IVT states that if the dolphin is below the water at a certain time and above the water one second later (or vice versa), then the dolphin had to have broken the surface within that one second.

Problem 4 *Consider the same function f as in Problem 1c. Using the IVT, prove that f is not continuous. (Hint: Argue by contradiction.)*

So far we have proven that certain functions are not continuous, but we have yet to show that a function is continuous. In fact this is much harder, because we must check every possible convergent sequence at every point. Fortunately, in most cases our intuitive Definition 1 will work, giving us several kinds of functions (some of which we encountered in Problem 1) which we know to be continuous: polynomials, e^x, $\sin(x)$, $\cos(x)$, their sums, their products, and their compositions (where $(f \circ g)(x) = f(g(x))$.

Problem 5 *a. Show that $f(x) = x^2 - x - 1$ has a zero in the open interval $(1, 2)$.*

b. Show that $f(x) = x^3 + \cos(x)$ has a zero in the open interval $(-1000000, 1000000)$. Can you find a better estimate of the zero (ie. a smaller interval which must contain it)?

c. Show that the graphs of $f(x) = x^{99} + \sin(x)$ and $g(x) = 1 - x$ intersect for some x in the open interval $(-2, 2)$ (Hint: The graphs intersect at a certain x if and only if $f(x) = g(x)$; can you rewrite that to an equation for which you can use IVT?)

Problem 6 *For each function below, prove or disprove if it has a zero anywhere on the real line, and if it does then give an interval where that zero occurs.*

a. $f(x) = x^2 + 1$

b. $f(x) = x^3 + x + 1$

c. $f(x) = e^x$

d. $f(x) = \log(|x| + \frac{1}{2})$ *(Hint: You can take for granted that this function is continuous.)*

To conclude this section we get to see one extremely powerful application of the IVT. Recall from fall quarter that we proved the Fundamental Theorem of Algebra; the IVT gives yet another proof.

Problem 7 *a. Prove that any polynomial with odd degree has at least one real root. (Hint: Go back to Problems 3b, 3c. What sort of pattern do we see there?)*

b. Prove that any quadratic polynomial has exactly two roots in the complex plane, up to multiplicity.

c. It is a known fact (which we may take for granted) that any polynomial of even degree factors into quadratics. Prove that any polynomial of degree $2n$ has exactly $2n$ complex roots up to multiplicity.

d. (The Fundamental Theorem of Algebra) Prove that any polynomial of degree n has exactly n complex roots up to multiplicity.

3 Bonus Section: Metric Spaces Other Than the Standard Real Line

So far we have been dealing with standard $\mathbb{R}$. Let us examine $\mathbb{R}^2$, with each of the four metrics we discussed last week: the Euclidean, taxicab, product, and discrete metrics.

Problem 8 *Define $f : \mathbb{R}^2 \to \mathbb{R}$ (with the standard metric on the latter) by*

$$f(x_1, x_2) = \begin{cases} 0 & x_1 < 0, x_2 > 0 \\ 1 & \textit{otherwise} \end{cases}$$

a. Put the Euclidean metric on the domain. Show that f is not continuous at $(0, 0)$.

b. Put the taxicab metric on the domain. Show that f is not continuous at $(0, 0)$.

c. Put the product metric on the domain. Show that f is not continuous at $(0, 0)$.

*d. Put the discrete metric on the domain. Show that f **is** continuous, in fact everywhere.*

(Hint: It may help to recall last week's handout.)

In addition to putting different metrics on the domain, we can also put different metrics on the range.

Problem 9 *Define $f : \mathbb{R} \to \mathbb{R}$, with the standard metric on the domain, by $f(x) = 2x$.*

a. Put the standard metric on the range. Show that f is continuous.

*b. Put the discrete metric on the range. Show that f is **not** continuous.*

For this problem, recall the definiton of *equivalent metrics* from last week.

Problem 10 *Let $f : X \to Y$.*

a. Fix a metric on Y, and let d and d' be equivalent metrics on X. Prove that f is continuous with respect to d if and only if it is continuous with respect to d'.

b. Fix a metric on X, and let d and d' be equivalent metrics on Y. Prove that f is continuous with respect to d if and only if it is continuous with respect to d'.