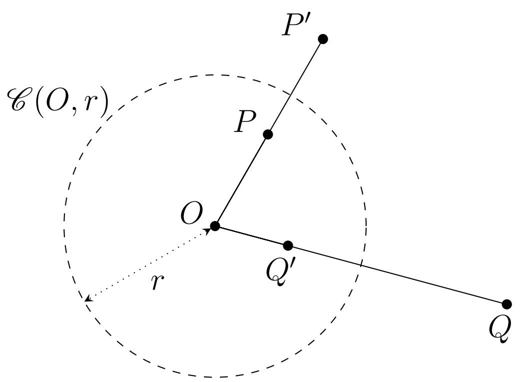
ORMC: INVERSION IN GEOMETRY

OLYMPIAD GROUP 1, WEEK 6

So far, we have only dealt with transformations that preserve classes of geometric objects; namely, translations, rotations, homotheties and reflections take *lines to lines* and take *circles to circles.*

Here is an example of a transformation that does not obey these rules, but is nevertheless very useful:



We fix a center point O, and a certain radius $r > 0$. We send each point $P \neq O$ to the point P' on the same line as O and P, with the property that $OP \cdot OP' = r^2$. This operation is called **inversion** around O with radius r, or more concisely **reflection across the circle** $\mathscr{C}(O, r)$.

Problem 1. (Elementary properties)

(a) What are the fixed points of the inversion? Does this correspond with the idea of "reflecting across a circle"?

(b) Where do points inside $\mathscr{C}(O, r)$ go? What about points outside? Does this correspond with the idea of "reflecting across a circle"?

(c) What happens if you apply the same inversion process twice in a row? Does this correspond with the idea of "reflecting across a circle"?

(d) Let P' be the reflection across $\mathscr{C}(O, r)$ of a point P inside the circle (as in the figure above), and let T be the intersection of PP' with the circle $\mathscr{C}(O, r)$. We keep P and T fixed and denote by d the distance between them. At the same time, we move O towards infinity by making r (and thus also $OP = r - d$) very large. What happens to the circle, and what point does P' approach? *Hint: Compute the distance TP' in terms of d and r, then let $r \to \infty$.*

Problem 2. (Composition properties)

(a) Show that in the complex plane with origin at O (the center of the circle), inversion with respect to the circle $\mathscr{C}(O, r)$ can be represented as the map $z \mapsto \frac{r^2}{\bar{z}}$.

(b) Show that the composition between a homothety and an inversion centered at the same point is still an inversion. Does it matter in which order we perform them?

(c) Show that the composition of two inversions centered at the same point is a homothety. What's the homothety factor; does the order matter?

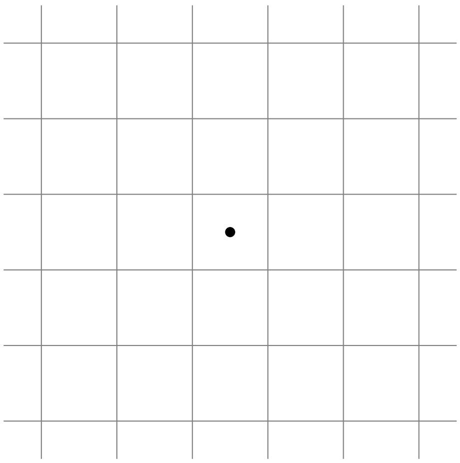
Problem 3. (Where do lines go?)

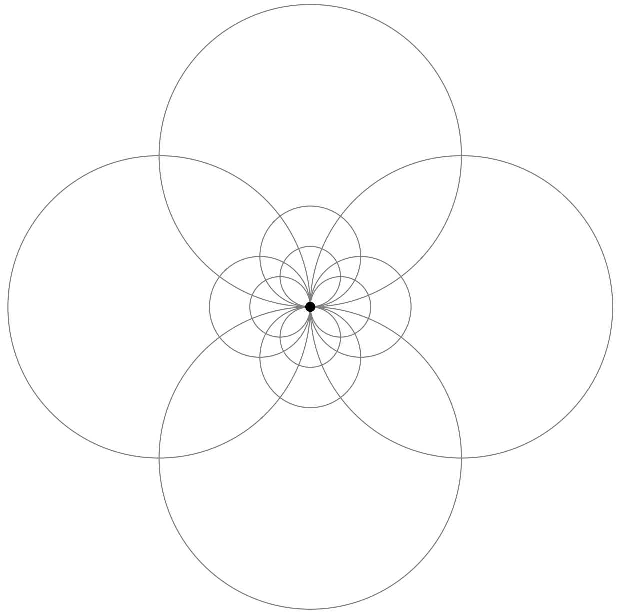
(a) Let $\triangle OHP$ be a triangle with $\hat{H} = 90^\circ$ and $OH = r$. Let P' be the foot of the perpendicular from H to OP. Show that $OP \cdot OP' = r^2$.

(b) Based on the previous part, where does a line *tangent* to $\mathscr{C}(O, r)$ go under inversion?

(c) How about a line *not necessarily* tangent to $\mathscr{C}(O, r)$? *Hints: you will have to consider whether O is on the line or not; Also, you should use the previous part, in conjunction with part (b) of Problem 2.*

(d) Where does a circle passing through O go? *Hint: use part (c) of Problem 1.*





Problem 4. (Where do circles go in general?)

(a) (Power of a point) Let $\mathscr{C}$ be a circle, and O be a point not on $\mathscr{C}$. Let ℓ be a line through O, intersecting $\mathscr{C}$ at points P, P' (which could be equal if ℓ is tangent). Show that the value $OP \cdot OP'$ is not dependent on the choice of ℓ; this is called the power of O with respect to $\mathscr{C}$, and we denote it by $P(O, \mathscr{C})$. *Warning: you should be careful about whether O is inside or outside $\mathscr{C}$. Technically speaking, the power is considered to be negative if O is inside $\mathscr{C}$, but this detail is not important for our purposes.*

(b) Assume O is outside $\mathscr{C}$. Where does the circle $\mathscr{C}$ go under an inversion centered at O of radius $\sqrt{P(O, \mathscr{C})}$?

(c) Assume O is inside $\mathscr{C}$. Where does the circle $\mathscr{C}$ go under an inversion centered at O of radius $\sqrt{-P(O, \mathscr{C})}$? *Hint: the difference is a sign.*

(d) Show that a circle not passing through O goes to a circle not passing through O, no matter what the radius of inversion is. *Hint: same technique as for the previous problem.*

Here is a summary:

Before	**After**
Line through O	Line through O
Line away from O	Circle through O
Circle through O	Line* away from O
Circle away from O	Circle away from O.

*This line is perpendicular to the diameter of the circle passing through O.

It quickly follows that tangency and parallelism also behave nicely with respect to inversion:

Before	**After**
Two circles tangent at O	Two parallel lines away from O
Two tangent circles away from O	Two tangent circles away from O
Two parallel lines away from O	Two circles tangent at O

Problem 5. (Showing that heights are concurrent)

(a) Let $\triangle ABC$ be an acute triangle, and let BB', CC' be heights ($B' \in AC$, $C' \in AB$). Show that there is an inversion centered at A, sending the circumcircle $(AC'B')$ to the line BC.

(b) Let H be the intersection of BB' and CC', and show that $H \in (AC'B')$, with AH a diameter.

(c) Using properties of inversion, show that $AH \perp BC$.

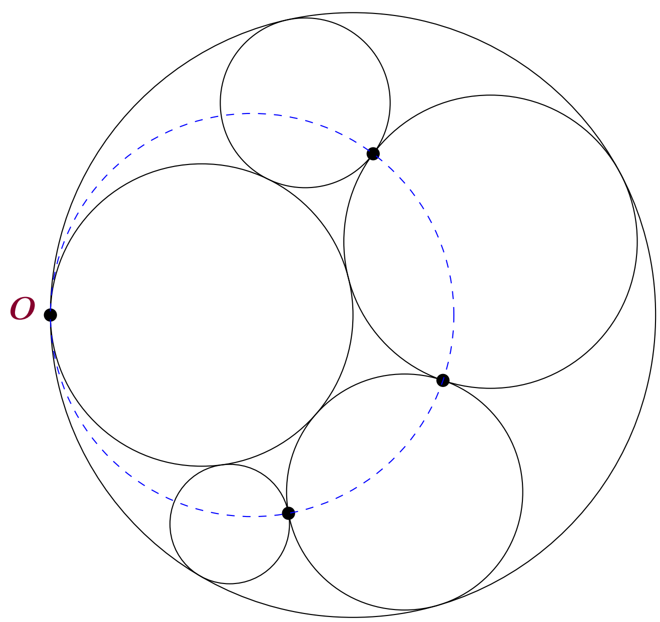
Problem 6. (USAMO 1993). Let $ABCD$ be a convex quadrilateral whose diagonals AC and BD are perpendicular and intersect at E. Prove that the reflections of E across AB, BC, CD and DA are concyclic.

Homework

Problem 1. Show that if we compose an inversion, a rotational homothety, and then another inversion with respect to the same center (in this order), we get a rotational homothety.

Hint: use part (a) of Problem 2, and recall that in the complex plane, a rotational homothety centered at 0 *has the form* $z \mapsto az$ *for some* $a \in \mathbb{C}\backslash\{0\}$.

Problem 2. We have 6 black circles tangent like in the picture below:



Show that the four marked tangency points all lie on a circle (the blue circle), as follows:

(a) Apply an inversion with center O and arbitrary radius. Show that the two black circles passing through O go to parallel lines (use that these circles are tangent at O).

(b) Where do the other black circles go? Draw a separate "inverted" figure.

(c) Show that the other three tangency points go to collinear points in the inverted figure. Conclude that the original four marked tangency points are concyclic.