Deja vu game

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1 Quadratic equations

1. Write down a cubic equation that has the roots $x_1 = 1$, $x_2 = 2$, and $x_3 = 3$.
2. Think of a quadratic equation with roots $5 \pm \sqrt{11}$.
3. Find the minimum of the function and x where the minimum is attained
$$f(x) = (x+1)^2 + x^2 + (x-3)^2 + (x-10)^2.$$
4. Find all the real solutions of the equation
$$7\left(x + \frac{1}{x}\right) - 2\left(x^2 + \frac{1}{x^2}\right) = 9$$
5. Let x_1 and x_2 be the roots of the equation $x^2 - 10x + 14$. Find $x_1^3 + x_2^3$.

2 Weighings, Logic, Invariants

1. Out of five coins, three coins are real and two are fake. Both fake coins weigh the same, but they weigh less than the real coins. What is the least number of times you have to use the balance scale to guarantee that you can find at least one real coin?

2. Out of 81 coins one is fake and 80 are genuine. A fake coin weighs less than a genuine coin. What is the least number of times you can use a balance scale to find the fake coin?

3. The numbers 1, 2, 3, ..., 99, 100 are written on the board. It is allowed to erase any two numbers a and b and write the number $a + b - 1$ instead. What number can remain on the board after 99 such operations?

4. Numbers from 1 to 2021 are written on the board. With one operation it is allowed to erase two numbers and write their positive difference instead. At some point there will be only one number left on the board. What is the minimal possible value of this number?

5. On the island of knights, knaves and spies, you come across three people. One wears blue, one wears red, and one wears green. You know that one is a knight, one is a knave, and one is a spy. "Who is the spy?" you ask.

   - The man wearing blue says, "That man in red is the spy."
   - The man wearing red says, "No, the man in green is the spy."
   - The man wearing green says, "No, the man in red is in fact the spy."

   Knights always tell the truth, knaves always lie, and spies do whatever they want.

   Who is the spy? Who is the knight and who is the knave?

3 Induction

1. Find out a formula for $1 + 2 + 3 + \cdots + n$ (no proof required).
2. Find out a formula for $1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + n \times (n + 1)$ (no proof required).
3. $x + \frac{1}{x} = 10$. Find $x^2 + \frac{1}{x^2}$, $x^3 + \frac{1}{x^3}$ and $x^4 + \frac{1}{x^4}$ (answer should include all 3 values).
4. Fibonacci numbers, commonly denoted F_n, form a sequence, called the Fibonacci sequence, such that each number is the sum of the two preceding ones, starting from 0 and 1. That is,

$$F_0 = 0,\ F_1 = 1,$$

and

$$F_n = F_{n-1} + F_{n-2},$$

for $n > 1$. How many numbers from F_0, F_1, F_2, $\ldots$, F_{2021} are even?

5. How many ways are there to split the 2×10 board into dominoes?

4 Geometry

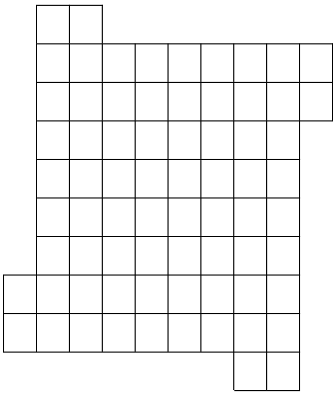
1. In a quadrilateral $ABCD$ angles ABC and ADC are right. Also, $\angle ABD = 40^\circ$. Find $\angle CAD$.

2. ℓ_1 and ℓ_2 are tangents to the circle ω touching the circle at the ends of a 200° arc. Find the angle between ℓ_1 and ℓ_2.

3. In a convex quadrilateral $ABCD$ one has $\angle DBC = 39^\circ$, $\angle ACD = 18^\circ$ and $\angle CDA = 123^\circ$. Find the angle ABC.

4. The center of the circle passing through the midpoints of the sides of triangle ABC, lies on the bisector of the angle A. What is the measure of angle A?

5. Cut the shape shown in the figure into 6 equal parts, making cuts along the sides of the cells.



5 Drawing contest

For that part you **don't** need to solve the problem. Just a drawing is enough

1. Prove that if through the tangency point of two circles two secants are drawn, then the chords connecting the endpoints of the secants are parallel.

2. From the intersection point of the diagonals of a rhombus, perpendiculars are dropped to the sides of the rhombus. Prove that the feet of these perpendiculars are vertices of a rectangle.

3. Prove that if one side and the median and the attitude drawn to that side in one triangle are respectively congruent to side and the median and the attitude drawn to that side in another triangle, then such triangles are congruent.

4. Let ABC be a triangle with $\angle C = 90°$, and A_0, B_0, C_0 be the midpoints of sides BC, CA, AB respectively. Two regular triangles AB_0C_1 and BA_0C_2 are constructed outside ABC. Find the angle $C_0C_1C_2$.

5. Consider the convex quadrilateral $ABCD$. The point P is in the interior of $ABCD$. The following ratio equalities hold:

$$\angle PAD : \angle PBA : \angle DPA = 1 : 2 : 3 = \angle CBP : \angle BAP : \angle BPC.$$

   Prove that the following three lines meet in a point: the internal bisectors of angles $\angle ADP$ and $\angle PCB$ and the perpendicular bisector of segment AB.