Metrics Revisited With Sequences

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1 Achilles and the Tortoise

In ancient Greek times, the warrior Achilles raced against a tortoise. Since it would obviously not be a fair race, he decided to give the tortoise a head start. Suppose that Achilles runs at 1 meter per second, the tortoise at 0.5 meters per second, and that the tortoise has a 1 meter head start. (In reality, they'd both be running quite a bit faster; we've adjusted the numbers to be more mathematically convenient)

The ancient Greek philosopher Zeno gave two different ways to think about this race.

Hypothesis 1 *We divide the race into steps:*

- *Step 1: Achilles runs for some time to catch up to where the tortoise started. Since this takes a positive amount of time, the tortoise has since moved past this point, so Achilles is behind.*
- *Step 2: Achilles runs for some more time to catch up to where the tortoise was after Step 1. Since this takes a positive amount of time from the end of Step 1, the tortoise has since moved past this point, so Achilles is still behind.*
- *Step 3: Achilles runs for some more time to catch up to where the tortoise was after Step 2. Since this takes a positive amount of time from the end of Step 2, the tortoise has since moved past this point, so Achilles is still behind.*
- *Rinse and repeat until the end of time.*

  The conclusion: Achilles will never catch the tortoise.

Hypothesis 2 *By subtracting their speeds, we find that Achilles is* 0.5 *meters per second faster than the tortoise. Therefore, after a long enough time (say,* 100 *seconds), Achilles will have covered* 50 *more meters than the tortoise which puts him ahead even after the tortoise's* 1 *meter head start.*

The conclusion: Achilles will overtake the tortoise.

The outcome of this race is known as *Zeno's paradox*, and the ancient Greeks struggled to find the logical flaw in either hypothesis. Zeno's solution was that all motion is an illusion, but since mathematics has taken many strides since Zeno's time, let's look at this problem again.

Problem 1 *Let us try to resolve Zeno's paradox.*

a. *Which hypothesis do we know to be correct? You can try racing a tortoise (or any slow-moving object) yourself if you are unsure.*

b. *Let us examine hypothesis 1 more closely. Is any step logically flawed? If so, can you correct it?*

c. *Is the conclusion that we are able to repeat steps until the end of time logically flawed? If so, can you correct it?*

You should have seen that when we split the race into steps, the total time elapsed after each step gets closer and closer to 2 seconds. In the early 19^{th} century, Cauchy formalized this as a *limit*, putting Zeno's paradox to rest once and for all.

2 Some Review

Intuitively, we can see that $1, 1\frac{1}{2}, 1\frac{3}{4}, 1\frac{7}{8}$ is getting close to 2. But what do we really mean by that? One way to make sense of "closeness" of two numbers is to somehow measure the distance between them. Recall that last quarter, we defined a way to write distance functions, or **metrics**.

Definition 1 *Given a nonempty set X, a* ***metric*** *on X is a function $d : X \times X \to \mathbb{R}$ such that*

i) $d(x, y) \geq 0$ for all $x, y \in X$.

ii) For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$.

iii) $d(x, y) = d(y, x)$ for all $x, y \in X$.

iv) (Triangle Inequality) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

(X, d) is called a ***metric space****.*

We also saw several examples of different metrics and proved they were indeed metrics:

- The **discrete metric** on any nonempty set X defined by

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

- The **Euclidean metric** on $\mathbb{R}^2$, defined by

$$d_E((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

- The **taxicab metric** on $\mathbb{R}^2$, defined by

$$d_T((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$$

- The **product metric** on $\mathbb{R}^2$, defined by

$$d((x_1, x_2), (y_1, y_2)) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$$

Though it's easiest to think of the last three examples in the plane $\mathbb{R}^2$ because we can draw it more easily, we can also extend each of these to higher and lower dimensions. In particular, these three metrics are all the same on the real line $\mathbb{R}$ and are called the **standard metric** $d(x, y) = |x - y|$.

Recall also the definition of an **open ball** in a metric space.

Definition 2 *Let (X, d) be a metric space. An* ***open ball*** *of radius $r > 0$ centered at $x_0 \in X$ is the set*

$$B(x_0, r) := \{x \in X \; : \; d(x_0, x) < r\}$$

Problem 2 *Draw or describe the following open balls.*

a. $B(0, r)$ (r is a radius $r > 0$) in $X = \mathbb{R}$ with the standard metric.

b. $B((0, 0), 1)$ in $X = \mathbb{R}^2$ with each of the Euclidean, taxicab, and product metrics.

c. $B((0, 0), 1)$ in $X = \mathbb{R}^2$ with the discrete metric.

d. $B((0, 0), 2)$ in $X = \mathbb{R}^2$ with the discrete metric.

3 Sequences and Convergence

Definition 3 *A **sequence** in a set X is a infinite collection $x_1, x_2, x_3, ...$ where $x_n \in X$ for all $n \in \mathbb{N}$. Denote it $(x_n)_{n=1}^{\infty}$.*

As an example, we consider the sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$, where

$$x_n = \frac{1}{n}$$
$$y_n = \begin{cases} 0 & n \text{ is even} \\ 1 & n \text{ is odd} \end{cases}$$

in $\mathbb{R}$ with the standard metric.

Problem 3 *a. Show that any open ball centered at 0 contains all but finitely many elements of x_n.*

b. Show that even though any open ball centered at 0 contains infinitely many elements of y_n, it may not contain all but finitely many elements of y_n.

Definition 4 *A sequence $(x_n)_{n=1}^{\infty}$ in a metric space X **converges** to $x \in X$ if every open ball centered at x contains all but finitely many elements of the sequence. We also say that x is a **limit** of the sequence $(x_n)_{n=1}^{\infty}$ and we write $x_n \to x$ or $x = \lim_{n\to\infty} x_n$ (This limit is not a priori unique, but it turns out to be.) †*

By Problem 3 we see that in the previous example, $x_n \to 0$, but $y_n \nrightarrow 0$, *under the standard metric on $\mathbb{R}$.*

Problem 4 *Let us return to the sequence $1, 1\frac{1}{2}, 1\frac{3}{4}, 1\frac{7}{8}, ...$ which we encountered in the very beginning. If we put the standard metric on $\mathbb{R}$, show that this sequence does in fact converge to 2. (This will rigorously prove your resolution to Zeno's paradox from Problem 1.)*

Problem 5 *Consider the sequence $((1/n, -1/n))_{n=1}^{\infty}$ in $\mathbb{R}^2$.*

a. Does this sequence converge under the Euclidean metric? If so, to what limit?

b. Does this sequence converge under the taxicab metric? If so, to what limit?

c. Does this sequence converge under the discrete metric? If so, to what limit?

d. Does this sequence converge under the product metric? If so, to what limit?

† Some astute observers may notice that this definition of convergence appears different from the one we learned at the start of the quarter. It is a good mathematical exercise to figure out, at least in the case with the standard metric on $\mathbb{R}$, why this definition is equivalent to the definition that $x_n \to x$ if for any $\epsilon > 0$, $|x_n - x| < \epsilon$ for all n sufficiently large.

4 Equivalent Metrics

From Problem 5 we see an example of a sequence which converges to the same limit in several different metrics. In fact this is not a coincidence, as although our notion of distance (the metric) may be different, our notion of "closeness" (convergence) may actually be the same.

Problem 6 a. *Show that any open Euclidean ball centered at any point $x \in \mathbb{R}^2$ contains an open taxicab ball centered at the same point x.*

b. *Show that any open taxicab ball centered at any point $x \in \mathbb{R}^2$ contains an open Euclidean ball centered at the same point x.*

c. *Show that $x_n \to x$ in the Euclidean metric if and only if $x_n \to x$ in the taxicab metric.*

Definition 5 *Two metrics d, d' are* ***equivalent*** *if they satisfy the properties outlined in Problem 6.*

Problem 7 *Show that the taxicab, Euclidean, and product metrics on $\mathbb{R}^2$ are all equivalent. (Hint: Using the previous problem, it only remains to show that the product metric is equivalent to either the taxicab or Euclidean metric. Do whichever is the most convenient for you.)*

Problem 8 a. *Given any metric d on any set X, define $d' : X \times X \to \mathbb{R}$ by*

$$d'(x, y) = \min\{d(x, y), 1\}$$

Is d' a metric? If so, is it equivalent to d?

b. *(Challenge) Given any metric d on any set X, define $d' : X \times X \to \mathbb{R}$ by*

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

Is d' a metric? If so, is it equivalent to d?