The largest power of six that does not exceed 450 is 216. Dividing 450 by 216, we have a quotient of 2 (and a remainder of 18). Thus the first digit of the numeral 450 for the base six system is **2**. Now we take the remainder 18 and divide it by the next smaller power of six—at the previous stage we divided by $6^3 = 216$, and now we divide by $6^2 = 36$. The quotient is 0, hence the second digit is **0**. The remainder is 18 and we divide this by the next smaller power of six; that is, by $6^1 = 6$. Now we see that the next digit is **3** (the remainder is 0). Therefore, the last digit (the quotient after the division by $6^0 = 1$) is 0. Finally, the base six representation of 450 is "2030".

While building our new system, we have not used any particular properties of the number 6, whatever they may be. Similarly, starting with any natural number n greater than 1, we can build a base n number system, in which the digits of a number are connected with its representation as a sum of powers of n. In this system, the number n is called the *base*. To avoid ambiguity, we will write the base of the system as a subscript (in decimal notation) at the right end of the numeral. Using this notation, we can rewrite the equalities indicated earlier as:

$$7_{10} = 11_6, 12_{10} = 20_6, 35_{10} = 55_6, 45_{10} = 113_6 \; .$$

Solve

Exercise 1. How many digit symbols do we need for a
 a) binary (that is, base 2) system;
 b) base n number system?

To write a number in the base n system, we must represent it in the following form:

$$a_k n^k + a_{k-1} n^{k-1} + \ldots + a_2 n^2 + a_1 n^1 + a_0 n^0 \; ,$$

where each a_i takes values from 0 to $n-1$, and a_k is not equal to zero (although the last restriction is, strictly speaking, not necessary).

Solve

Exercise 2. Write in decimal notation the numbers 10101_2, 10101_3, 211_4, 126_7, and 158_{11}.

Solve

Exercise 3. Write the number 100_{10} in the systems with bases 2, 3, 4, 5, 6, 7, 8, and 9.

Solve

Exercise 4. In a system whose base is greater than 10, we need more than ten digit symbols, so we must invent some. For example, in the base 11 system, we might use "A" to represent the "digit" 10. So, for example, 21_{10} could be written as 1A. Using this convention, write the number 111_{10} in the base eleven notation.

Let us learn how to add and multiply numbers written in an arbitrary system. We can do this in exactly the same way as in the decimal system, but we must remember that a "carry" occurs each time the result of adding up digits in a column exceeds or equals the base of the given number system.

Below is an example of the addition of the two numbers 124_{10} and 417_{10} in the base 3 system. First, we rewrite the numbers in the base 3 system: $124_{10} = 11121_3$, $417_{10} = 120110_3$. Then we write them one under another, lining up their rightmost digits. "Carries" are given in the upper row in small print.

$$\begin{array}{rcccccc}
 & & {\scriptstyle (1)} & & {\scriptstyle (1)} & {\scriptstyle (1)} & \\
 & & 1 & 1 & 1 & 2 & 1 \\
+ & 1 & 2 & 0 & 1 & 1 & 0 \\
\hline
 & 2 & 0 & 2 & 0 & 0 & 1
\end{array}$$