CHAPTER 15

Number Bases

§1. What are they?

Any student can say that "2653" stands for the number "two thousand six hundred fifty three", whatever that may mean. How do they know this? We are all accustomed to the following way of writing numbers: the last digit denotes the number of units in the given number, the next-to-last—the number of tens, the third last—the number of hundreds, and so on (though this is a bit ambiguous, since the number of units in 2653 is, in a way, not 3, but 2653!). This way of writing numbers (and interpreting strings of digits) is called in brief a *number base system.* Thus, writing "2653", we think of the number $\mathbf{2}\cdot 1000+\mathbf{6}\cdot 100+\mathbf{5}\cdot 10+\mathbf{3}\cdot 1$, or shortly, $\mathbf{2}\cdot 10^3+\mathbf{6}\cdot 10^2+\mathbf{5}\cdot 10^1+\mathbf{3}\cdot 10^0$. We print the digits of the number in boldface to make it easier to distinguish them from other numbers.

We can easily see that the number ten plays a special part in this representation: any other number is written as a sum of different powers of ten with coefficients taking values 0 through 9. This is why this system is called "decimal" (from the latin word for "ten"). To write a number we use the ten special symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9, called digits. They denote the numbers from zero to nine. The next number, that is, ten, is regarded as a unit of the next level and is written with two digits: 10, which, roughly speaking, means "add up one times ten and zero times one".

Now, what if we used some other number, say, six? Analogously, we would need six symbols as digits. We can take the six familiar symbols 0, 1, 2, 3, 4, and 5, which will denote the numbers from zero to five. The number six will be the unit of the next level, and, therefore, it will be written as 10. Proceeding with this analogy, we can represent each natural number as the sum of different powers of six with coefficients from 0 to 5. For instance (all numbers are written in the decimal system):

$$
\begin{aligned}
7 &= \mathbf{1}\cdot 6^1+\mathbf{1}\cdot 6^0,\\
12 &= \mathbf{2}\cdot 6^1+\mathbf{0}\cdot 6^0,\\
35 &= \mathbf{5}\cdot 6^1+\mathbf{5}\cdot 6^0,\\
45 &= \mathbf{1}\cdot 6^2+\mathbf{1}\cdot 6^1+\mathbf{3}\cdot 6^0.
\end{aligned}
$$

Thus in our new number system (which is called the "base six system") we write the number 7 as "11", the number 12 as "20", 35 as "55", and 45 as "113".

It is easy to see that we can write any natural number in the base six system. We show how to do this for the number 450 (in this example, as earlier, all the given numbers are written in the decimal system unless enclosed in quotes).