Lesson 5: Stars and Bars II

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Problem 1.

a) Show that

\[ 1 + 2 \binom{n}{1} + 2^2 \binom{n}{2} + \ldots + 2^n \binom{n}{n} = 3^n \]

b) Show that

\[ 2^{2n} \binom{2n}{2n} - 2^{2n-1} \binom{2n}{2n-1} + \ldots + 2^2 \binom{2n}{2} - 2 \binom{2n}{1} = 0 \]

Problem 2.

Show that the number of ways to write a positive integer \( n \) as a sum of \( k \) nonnegative numbers is

\[ \binom{n+k-1}{k-1} \]

Problem 3.

At a math circle meeting 10 students are solving 10 problems. Any two students solved a different number of problems, and every problem is solved by the same number of students. Viserion solved problems 1 through 5 and did not solve problems 6 through 9. Did he solve problem 10?

Problem 4.

A toy consists of a ring with 3 red beads and 7 blue beads on it. If two configurations of beads differ only by rotations and reflections, they are considered the same toy. How many different toys are there?

Problem 5.

Let \( K \) be a point on the side \( CD \) of a square \( ABCD \) such that \( CK/KD = 1/2 \). If the side length of the square is 1, find the length of the perpendicular from \( C \) to the line \( AK \). You may use the Pythagorean theorem.

Problem 6.

Let \( AA_1, BB_1 \) and \( CC_1 \) be the altitudes of the acute triangle \( \triangle ABC \). Show that if \( A_1B_1 \parallel AB \) and \( B_1C_1 \parallel BC \), then \( A_1C_1 \parallel AC \).