Lesson 5: Stars and Bars II

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Problem 1.
a) Show that

$$1 + 2\binom{n}{1} + 2^2\binom{n}{2} + \ldots + 2^n\binom{n}{n} = 3^n$$

b) Show that

$$2^{2n}\binom{2n}{2n} - 2^{2n-1}\binom{2n}{2n-1} + \ldots + 2^2\binom{2n}{2} - 2\binom{2n}{1} = 0$$

Problem 2.
Show that the number of ways to write a positive integer n as a sum of k nonnegative numbers is

$$\binom{n+k-1}{k-1}$$

Problem 3.
At a math circle meeting 10 students are solving 10 problems. Any two students solved a different number of problems, and every problem is solved by the same number of students. Viserion solved problems 1 through 5 and did not solve problems 6 through 9. Did he solve problem 10?

Problem 4.
A toy consists of a ring with 3 red beads and 7 blue beads on it. If two configurations of beads differ only by rotations and reflections, they are considered the same toy. How many different toys are there?

Problem 5.
Let K be a point on the side CD of a square $ABCD$ such that $CK/KD = 1/2$. If the side length of the square is 1, find the length of the perpendicular from C to the line AK. You may use the Pythagorean theorem.

Problem 6.
Let AA_1, BB_1 and CC_1 be the altitudes of the acute triangle $\triangle ABC$. Show that if $A_1B_1 \parallel AB$ and $B_1C_1 \parallel BC$, then $A_1C_1 \parallel AC$.