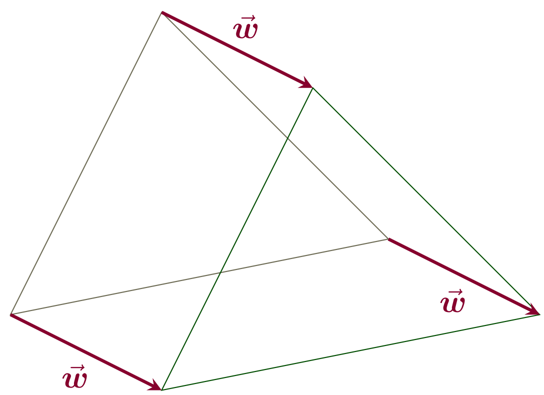
ORMC: GEOMETRIC TRANSFORMATIONS

OLYMPIAD GROUP 1, WEEK 5

For our purposes, a geometric transformation is a map T from the Euclidean plane to itself. It's often helpful to envision such transformations as "moving" each point A to its image point $T(A)$; we'll later look at images of entire sets of points (such as lines or triangles).

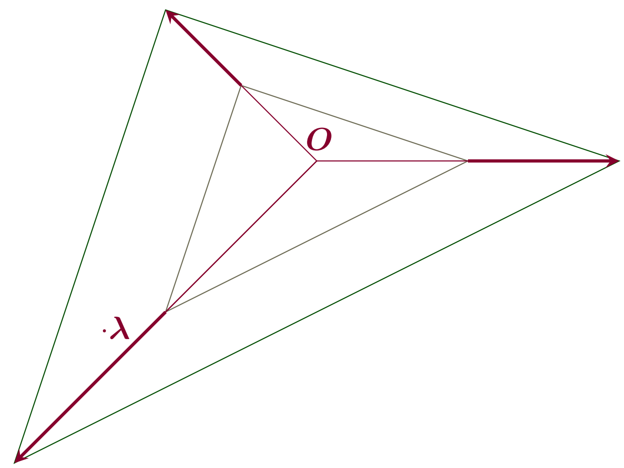
- Translations (by a translation vector $\vec{w}$):



In the complex plane:

$$z \longmapsto z + w$$

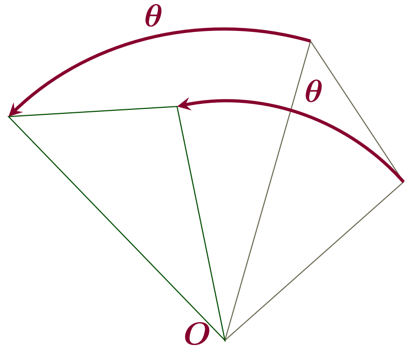
- Homotheties or dilations (with respect to a center O, by a *dilation factor* of λ):



In the complex plane (origin $= O$):

$$z \longmapsto \lambda \cdot z$$

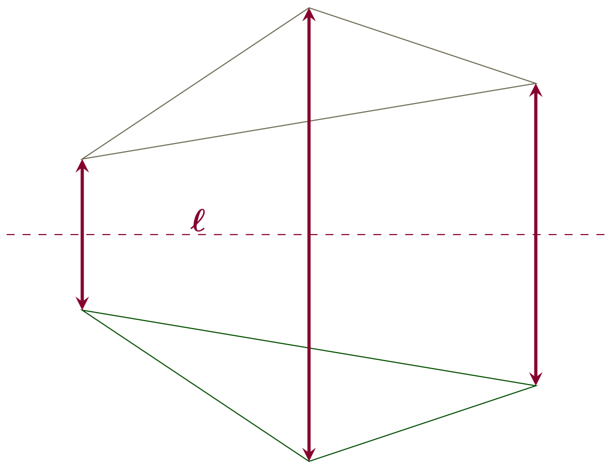
- Rotations (by a counterclockwise angle θ rad, with respect to a *center of rotation* O):



In the complex plane (origin $= O$):

$$z \longmapsto z \cdot e^{i\theta} = z(\cos\theta + i\sin\theta)$$

- Reflections (with respect to a line ℓ):



In the complex plane (real axis $= \ell$):

$$z \longmapsto \overline{z} = \mathrm{Re}(z) - i\,\mathrm{Im}(z)$$

[*Recap: Absolute values, arguments and conjugation.*]

More generally, if $\lambda \neq 0$ and z_0 are complex numbers, then the map

$$\begin{aligned} z &\longmapsto \lambda(z - z_0) + z_0 \\ &= \lambda z + z_0(1-\lambda) \end{aligned}$$

is a **rotational homothety** (a rotation plus a homothety with the same center) centered at z_0, with dilation factor $|\lambda|$ and angle $\arg \lambda$. Hence if w is a complex number,

$$z \longmapsto \lambda z + w \quad \text{is} \quad \begin{cases} \text{a translation by } w, & \text{if } \lambda = 1, \\ \text{a rotational homothety centered at } \frac{w}{1-\lambda}, & \text{if } \lambda \neq 1. \end{cases}$$

Similarly, the **reflection with respect to a line at angle θ with the real axis** can be represented as

$$z \longmapsto e^{2i\theta}\overline{z} + w, \qquad \text{where } w \text{ is the reflection of the origin,} \tag{1}$$

so all compositions of the transformations that we are concerned with today can be represented as $z \longmapsto \lambda z + w$ or as $z \longmapsto \lambda \overline{z} + w$ for some λ and w.

Problem 1. (Preserving properties) Prove the following using complex numbers:

(a) A nontrivial rotational homothety preserves exactly one point (which one is it?). Does a translation preserve any point (i.e., keep it fixed)? How about reflections?

(b*) Translations, rotations, homotheties and reflections map lines to lines, and circles to circles. *Hint:* a line consists of points $\{z = at + b : t \in \mathbb{R}\}$ for some a and b, while a circle consists of points $\{z : |z - z_0| = r\}$, for some z_0 and r.

(c*) Translations, rotations and homotheties preserve oriented angles, and reflections preserve unoriented angles.

(d*) Translations, rotations and reflections preserve lengths of segments, while homotheties dilate lengths by the dilation factor $|\lambda|$. How would you guess that homotheties affect areas?

Problem 2. (Tangent circles) Two circles are externally tangent at P. Let AB be a line through P such that A belongs to the first circle and B to the second one (none of them being P). Show that the tangent at A to the first circle and the tangent at B to the second circle are parallel.

Hint: First, reflect the figure with respect to the line of the centers; what happens to the circles and why? Next, consider a homothety with negative factor centered at P; use that there's a unique circle passing through three points.

Problem 3. (Compositions) Prove the following using complex numbers:

(a) Two rotational homotheties with the same center *commute*, i.e. the order in which we perform them does not matter.

(b) The composition of a rotational homothety and a translation is a rotational homothety.

(c) The composition of any two reflections is either a rotation or a translation. When is it a translation?

Problem 4. (Two homotheties) Let $ABCD$ be a convex quadrilateral, and let X be the intersection of its diagonals. Let $E \in DB$ such that $AE \parallel CD$, and let $F \in AC$ such that $DF \parallel AB$. Prove that $EF \parallel BC$.

Problem 5. (Two reflections) Let $ABCD$ be a square, and let X and Y be points on sides CD and BC respectively, such that $\angle XAY = \angle DAX + \angle YAB$. Let C' be the reflection of C with respect to AX, and let C'' be the reflection of C' with respect to AY. Show that B is the midpoint of CC''.

Hint: The composition of these two reflections cannot be a translation (why?), so it must be a rotation (with what center?). Look at what D maps to under this rotation to determine the rotation's angle.

Problem 6. (Similar triangles)

(a) Let $\{A, B\}$ and $\{A', B'\}$ be pairs of distinct points in the plane. Using complex numbers, show that there is a unique rotational homothety or translation mapping $A \mapsto A'$ and $B \mapsto B'$.

(b) Let $\triangle ABC$ and $\triangle A'B'C'$ be similar triangles oriented alike. Show that there is a unique rotational homothety or translation mapping $A \mapsto A'$, $B \mapsto B'$, $C \mapsto C'$ (and thus $\triangle ABC \mapsto \triangle A'B'C'$). *Hint: Use part (a) and Problem 1.*

Problem 7. (Centroid) Let $\triangle ABC$ be a triangle, and let A', B', C' be the midpoints of BC, CA and AB respectively. Note that $\triangle ABC$ and $\triangle A'B'C'$ are similar and oriented alike, and consider the corresponding rotational homothety mapping $A \mapsto A'$, $B \mapsto B'$, $C \mapsto C'$. What must its angle be, given that $BC \parallel B'C'$? What about its dilation factor? What does this imply about its center?

Problem 8. (Euler's circle and line) Let $\triangle ABC$ be an acute, scalene triangle, and let H be its orthocenter (i.e. the intersection of all three heights).

(a) Show that the reflection of H across the line BC, as well as the reflection of H across the midpoint of BC are both on the circumcircle (ABC).

(b) Let M_A, M_B, M_C be the midpoints of the segments BC, CA, AB respectively. Let H_A, H_B, H_C be the feet of the heights from A, B and C respectively (so that AH_A, BH_B, CH_C intersect at H). Finally, let N_A be the midpoint of AH, and similarly N_B and N_C. Show that M_A, M_B, M_C, H_A, H_B, H_C, N_A, N_B, N_C all lie on a circle, which is called Euler's nine-point circle. *Hint: Perform a positive homothety centered at H with dilation factor $\lambda = 1/2$, and show that the circumcircle (ABC) is mapped to this Nine-point circle.*

(c) By part (b) of this exercise, we have shown that (ABC) is mapped to $(M_AM_BM_C)$ by a homothety centered at H. On the other hand, in problem 7, we have seen that $\triangle ABC$ is mapped to $\triangle M_AM_BM_C$ through a homothety centered at G, the centroid. Thus, there exist homotheties centered at both G and H that take (ABC) into the nine-point circle. Use this to show that the centers of both (ABC) and the nine-point circle lie on the line HG. This is called Euler's line.

Problem 9. (Toricelli's point). Let $\triangle ABC$ be an acute triangle (or, if you want to be precise, with angles less than 120°.) Find a way to construct the point T inside $\triangle ABC$ which minimizes the sum $TA+TB+TC$ (assume such a point indeed exists). *Hint: Perform a* $60°$ *rotation centered at* B*, sending* $A \longmapsto A'$ *and* $T \longmapsto T'$*, and show that* $TA+TB+TC = A'T' + T'T + TC$*. Then use the triangle inequality.*

Homework 1

Problem 1. (Rotation in Cartesian coordinates) Consider a counter-clockwise rotation centered at the origin $(0,0)$, of angle θ. Show that a point of coordinates (x,y) goes to the point $(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$. Then, plug in $\theta = 90°$, and see what you get. The angle $45°$ is also interesting.

Hint: The point (x,y) *corresponds to the complex number* $x+iy$*, and the rotation by* θ *to multiplication by* $e^{i\theta} = \cos\theta + i\sin\theta$*.*

Problem 2. Consider two distinct lines ℓ_1, ℓ_2 that intersect at a point O. Show that the reflections along ℓ_1 and ℓ_2 commute (i.e. the order in which you perform them does not matter) if and only if $\ell_1 \perp \ell_2$.

Hint: Use complex numbers and pick the origin (the complex number 0*) to be at* O*. You can also pick the real line to be* ℓ_1 *to make things easier, and let* θ *be the counterclockwise angle from* ℓ_1 *to* ℓ_2*. What do the reflections along* ℓ_1 *and* ℓ_2 *look like as complex functions (check out equation (1)), and what happens when you compose them in either order?*