THIS IS FUN! SPRING EDITION

OLGA RADKO MATH CIRCLE
ADVANCED 2
MAY 9TH, 2021

1. Measurement Errors

Problem 1. Each part is worth 10 points, and the last question is worth 20 for a total of 50

You set out to measure the gravity in your home in the most inefficient way:

(1) You set up two platforms, one on top of the other. You use a ruler to measure the distance between these platforms. You measure 1.76 meters. **If h is the height between the platforms, what is $\Delta(h)$?**

(2) You set a bowling ball at the top platform, and rig the platforms in the following way. When the top platform is removed, a rubber plug is released from a water tank and water flows out at a constant rate into a separate container. When the bowling ball hits the bottom platform, the plug reconnects and the flow of water stops.
   - To measure the rate at which water flows out, you release the plug for 60 seconds (looking at an analog clock) and you find that the volume of water released is 3.7 liters. **Let r be the flow rate in liters per second. Find r and $\Delta(r)$.**

(3) You remove the top platform, the bowling ball drops, hits the bottom platform and you weigh your container of water to see how much flowed out. Your scale reads 36.8 grams. **Let t be the time it took for the ball to drop from the top platform to the bottom platform. Find t and $\Delta(t)$.**

Use the above measurements to estimate the gravitational constant, g, in your home. Calculate the absolute error, $\Delta(g)$.

It might be useful to know that 1 liter of water weighs 1000 grams. You will also need the formula $y_f - y_i = -\frac{1}{2}gt^2$, where y_f is the final height, y_i is the initial height, g is the gravitational constant, and t is the time elapsed.

2. Lattices and Numbers

Problem 2. (Each part is worth 5 points)
Which of the following sets generate the lattice $\mathbb{Z}^n$ for their respective n? You need to be able to explain your reasoning.

(1) {(3,1),(-1,0)}
(2) {(1,0),(0,-1),(2,1)}
(3) {(1,1,0),(1,0,1),(0,1,1)}
(4) {(4,1), (1,4)}

Problem 3. (10 points) A **Pythagorean triple** consists of three **integers** a, b, c such that $a^2+b^2=c^2$. Describe what a Pythagorean triple corresponds to geometrically in terms of lattice points and a circle.

Problem 4. (10 points) Use the expression $(m^2+n^2)^2-(m^2-n^2)^2$ to show that there are infinitely many Pythagorean triples where a, b, c are relatively prime (don't share any common factors).

3. Smart Codes

Problem 5. (10 points) Which of the following 10-digit ISBNs are valid?

(a) 0679731725
(b) 0425164349
(c) 1485033192

Problem 6. (5 points each) Find the missing digit (the question mark) in each of the following valid 13-digit ISBNs.

(a) 8239**?**84912742
(b) 314159265358**?**
(c) 112**?**581321345

Problem 7. (10 points) Which of the following 13-digit ISBNs could have undetectable transposition error?

(a) 9781402894626
(b) 9789295055025
(c) 9780143145158

Problem 8. (5 points each) For the following Hamming's square codewords, correct single-digit or transposition errors.

(a) 011101000
(b) 101011110
(c) 100011101

Problem 9. (10 points) Convert the decimal number 9 to binary. Encode the 4-bit block using Hamming's [7,4]-code.

4. Miscellanous

Problem 10. (20 points max) Three pirates have successfully stolen 10 coins and must split the winnings. In each round of discussion, the eldest pirate proposes a split, which is voted on. If the vote passes with at least 50% approval, the split is enacted, otherwise the pirate who proposed the split is thrown overboard and the discussion proceeds to the next round. The pirates are selfish, and they vote "no" if they don't care.

As the eldest pirate, propose a way to split the 10 coins. When you submit your answer, two instructors will be your crewmates. If your proposed split is passed, the number of coins you proposed for yourself will be your score.

You can play this twice.

Problem 11. (15 points max) Two piles of 10 coins each are on the table. Taking turns, you and an opponent choose one pile and remove either 1 or 2 coins from the pile. The person who removes the last coin wins.

When you submit your answer, you will play against an instructor three times. In each time, win the game for 5 points, for a total of up to 15 points. In the first time, you will play second. In the second and third times, you will play first, but the instructor will make a mistake. You cannot retry this problem.

Problem 12. (20 points max) There are 12 visually identical coins labeled $1, \ldots, 12$, but one has a different weight than the other 11, which have identical weight. (You don't know if it's heavier or lighter.) You also have a scale that can tell you which of two groups of coins is heavier or lighter, which you can use up to 3 times.

Determine which coin is different. When you submit your answer, the instructor will have 4 sets of predetermined coins (for fairness between teams). For each set of coins, you can ask the instructor to weigh two groups against each other up to 3 times, and each time the instructor will respond "first group heavier," "second group heavier," or "same." After weighing 3 times, you will be asked to guess which coin is different, and how it is different. Each correct guess gives 5 points, for a maximum of 20 points total. You cannot retry this problem.

Problem 13. (10 points) Simplify the following product:

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{225}\right).$$

Problem 14. (10 points) The set $\{3, 6, 9, 10\}$ is augmented by a fifth element n, not equal to any of the other four. The median of the resulting set is equal to its mean. What is the sum of all possible values of n?

Problem 15. (10 points) Call a set of integers spacey if it contains no more than one out of any three consecutive integers. How many subsets of $\{1, 2, 3, \ldots, 12\}$, including the empty set, are spacey?

Problem 16. (10 points) One hundred people line up to board an airplane. Alice, the first passenger to board, lost her boarding pass and takes a random seat instead. Each subsequent passenger takes his or her assigned seat if available, otherwise a random unoccupied seat. What is the probability that Zachariah, the last passenger to board, finds his seat occupied?

Problem 17. (10 points) How many natural numbers n, less than 10000, for which $2^n - n^2$ is divisible by 7?

Problem 18. (10 points) The Mayor of Los Angeles is planning to split the city into 27 districts, such that each district shares a border with exactly 5 other districts. A cartographer tried and failed to draw a map that fulfills the Mayor's request. Can you help the cartographer and show that such a map cannot exist?

Problem 19. (20 points) Prove that there are infinitely many primes p such that $p \equiv 3 \pmod 4$. (Hint: Suppose to the contrary that there are finitely many such primes, find a contradiction.)

Problem 20. (20 points) How many non-congruent right triangles with positive integer leg lengths have areas that are numerically equal to 3 times their perimeters?