UCLA Math Circle: Projective Geometry

Yingkun Li
11/20/2011

This handout follows the §9 of [2] and §1 of [1]. For further information, you can check out those two wonderful books.

1 Homogeneous Coordinate

On the real number line, a point is represented by a real number x if x is finite. Infinity in this setting could be represented by the symbol ∞. However, it is more difficult to calculate with infinity. One way to resolve this issue is through a different coordinate system call *homogeneous coordinate*. For the real line, this looks a lot like fractions. Instead of using one number $x \in \mathbb{R}$, we will use two real numbers $[x : y]$, with at least one nonzero, to represent a point on the number line. Furthermore, we define the following equivalent relationship $\sim$

$$[x : y] \sim [x' : y'] \Leftrightarrow \text{ there exists } \lambda \in \mathbb{R}\backslash\{0\} \text{ s.t. } \lambda x = x' \text{ and } \lambda y = y'.$$

Now two points $[x : y]$ and $[x' : y']$ represent the same point if and only if they are equivalent. The set of coordinates $\{[x : y] \mid x, y \in \mathbb{R} \text{ not both zero}\}$ with the equivalence relation $\sim$ represents all the points on the real number line including ∞, i.e. $[1 : 0]$. Furthermore, the operation on the real number line, such as addition, multiplication, inverse, can still be done in this new coordinate system. The line described by this coordinate system is called the *projective line*, denoted by the symbol $\mathbb{RP}^1$.

Now it is not too difficult to extend this coordinate system to the plane. Instead of using two real numbers as coordinate, we will use three real numbers $[x : y : z]$, not all zero, with similar equivalence relationship to represent points in the plane

$$[x : y : z] \sim [x' : y' : z'] \Leftrightarrow \text{ there exists } \lambda \in \mathbb{R}\backslash\{0\} \text{ s.t. } \lambda x = x', \lambda y = y' \text{ and } \lambda z = z'.$$

The plane obtained under this coordinate system is called the *projective plane*, denoted by the symbol $\mathbb{RP}^2$. The finite points in the real plane can be embedded into the projective plane via the following map

$$i : \mathbb{R}^2 \rightarrow \mathbb{RP}^2 \quad (1)$$
$$(x, y) \mapsto [x : y : 1]. \quad (2)$$

Besides the points $i(\mathbb{R}^2)$, $\mathbb{RP}^2$ contains all the points in $\mathbb{RP}^2$ with the third coordinate zero. These form exactly a projective line. So $\mathbb{RP}^2$ can be obtained from $\mathbb{R}^2$ by adding an $\mathbb{RP}^1$ at infinity. Alternatively, one can create the projective plane from $\mathbb{R}^3$ by removing the points $(0, 0, 0)$ and identify points with the same direction.

Given a polynomial $f(x, y)$ on $\mathbb{R}^2$ in variable x, y, one can make it into a polynomial $F(X, Y, Z)$ in the variable X, Y, Z by setting

$$F(X, Y, Z) = Z^d F\left(\frac{X}{Z}, \frac{Y}{Z}\right), \tag{3}$$

where d is the degree of $f(x, y)$. This process yields a polynomial which satisfies the property

$$F(\lambda X, \lambda Y, \lambda Z) = \lambda^d F(X, Y, Z). \tag{4}$$

Any polynomial satisfying (4) is called a *homogeneous polynomial*. If $F(X, Y, Z)$ is a homogeneous polynomial, then the zero set of $F(X, Y, Z)$ in $\mathbb{R}^3$, i.e. the following set

$$Z(F) := \{(x_0, y_0, z_0) \in \mathbb{R}^3 \backslash \{(0, 0, 0)\} \mid F(x_0, y_0, z_0) = 0\} \tag{5}$$

can be considered as points in $\mathbb{RP}^2$. For different polynomials F, the set of points $Z(F)$ describes different geometric shapes in $\mathbb{R}^2$ that we are all familiar with.

For example, this coordinate gives us a simple and unified representation of lines. In the xy-coordinate plane, there are several ways to represent a line, i.e. point-slope form, slope-intercept form, etc. In $\mathbb{RP}^2$, however, a line is represented by one equation

$$aX + bY + cZ = 0, \tag{6}$$

with fixed $a, b, c \in \mathbb{R}$, not all zero. The line does not change if the set of parameters $\{a, b, c\}$ with $\{\lambda a, \lambda b, \lambda c\}$ for some nonzero real number λ. So a line is really represented by a point $[a : b : c] \in \mathbb{RP}^2$. Conversely, a point in $\mathbb{RP}^2$ represents a line in $\mathbb{RP}^2$. Thus, we have the following duality

$$\begin{aligned} \{\text{Points in } \mathbb{RP}^2\} &\leftrightarrow \{\text{lines in } \mathbb{RP}^2\} \\ [a : b : c] &\leftrightarrow aX + bY + cZ = 0. \end{aligned}$$

Alternatively, if you consider $[x : y : z]$ as points in $\mathbb{R}^3 \backslash \{(0, 0, 0)\}$, then it satisfies (6) if and only if it is perpendicular to the vector (a, b, c).

2 Pappus's Theorem and Desargues's Theorem

Now using homogeneous coordinate, we can give a criterion for collinearity of points. Let $P_i = [a_i : b_i : c_i]$, for $i = 1, 2, 3$, be three points in $\mathbb{RP}^2$. Then they are collinear if and only if the following determinant is 0

$$\det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}. \tag{7}$$

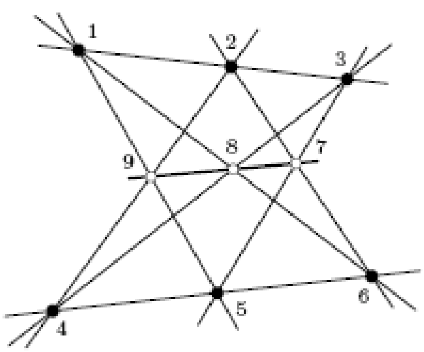
If this is the case, then the line passing through all three points is represented by $[u : v : w]$, where

$$ua_i + vb_i + wc_i = 0, \text{ for all } i = 1, 2, 3. \tag{8}$$

Using this, we can give a proof of Pappus's hexagon theorem (see [2] §1.3).

Theorem 1 (Pappus's Hexagon Theorem)**.** *Let P_1, P_2, P_3 be three collinear points on line ℓ and P_4, P_5, P_6 be three collinear points on line m. Suppose ℓ and m are distinct and P_i are distinct points. Then the following three points are collinear*

$$P_7 := \text{ intersection of } P_2P_6 \text{ and } P_3P_5, \tag{9}$$
$$P_8 := \text{ intersection of } P_1P_6 \text{ and } P_3P_4, \tag{10}$$
$$P_9 := \text{ intersection of } P_2P_4 \text{ and } P_1P_5. \tag{11}$$
$$\tag{12}$$



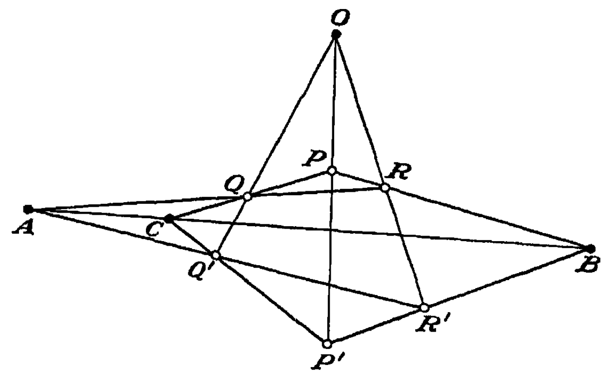
1	1	0	0
2	a	b	c
3	d	e	f
4	0	1	0
5	g	h	i
6	j	k	l
7	0	0	1
8	m	n	o
9	p	q	r

$$[1,2,3] = 0 \Longrightarrow ce = bf$$
$$[1,5,9] = 0 \Longrightarrow iq = hr$$
$$[1,6,8] = 0 \Longrightarrow ko = ln$$
$$[2,4,9] = 0 \Longrightarrow ar = cp$$
$$[2,6,7] = 0 \Longrightarrow bj = ak$$
$$[3,4,8] = 0 \Longrightarrow fm = do$$
$$[3,5,7] = 0 \Longrightarrow dh = eg$$
$$[4,5,6] = 0 \Longrightarrow gl = ij$$
$$[7,8,9] = 0 \Longleftarrow mq = np$$

Proof.
Consider the figure above. We can embed it in $\mathbb{RP}^2$ such that P_1 has coordinate $[1 : 0 : 0]$, P_4 has coordinate $[0 : 1 : 0]$ and P_7 has coordinate $[0 : 0 : 1]$. Let the rest of the points have the coordinates above. Let $[i, j, k] = 0$ denote points P_i, P_j and P_k are collinear. Since none of the points is collinear with P_1 and P_4, the third coordinate of every point, other than P_1 and P_4, is nonzero. Similar argument using P_1, P_7 and P_4, P_7 shows that all the coordinates of $P_2, P_3, P_5, P_6, P_8, P_9$ are nonzero. Finally the deduction on the right gives us the desired result □

Another nice theorem about collinearity is Desargues's Two-triangle Theorem. It can be deduced from Pappus's theorem. We will leave the proof as an exercise. One can consult §1-5 in [1] for more information.

Theorem 2. *If two triangles have corresponding vertices joined by concurrent lines, then the intersections of the corresponding sides are collinear.*



3 Parabolas in the Projective Plane

In $\mathbb{R}^2$, parabolas are usually represented by a quadratic equation $y = ax^2 + bx + c$. However, this is only a special case when the directrix is parallel to the x-axis. In general, the equation of a conic section, in particular a parabola, is quadratic in both x and y.

Using the homogenizing process before, we can change the equation of a conic section to three variables X, Y, Z. For example, here are some equations in $\mathbb{R}^2$ and $\mathbb{RP}^2$

	Equations in $\mathbb{R}^2$	Equations in $\mathbb{RP}^2$
line:	$ax + by + c = 0$	$aX + bY + cZ = 0$
circle:	$x^2 + y^2 = 1$	$X^2 + Y^2 - Z^2 = 0$
parabola:	$y = x^2$	$X^2 - YZ = 0$
hyperbola 1:	$xy = 1$	$XY - Z^2 = 0$
hyperbola 2:	$x^2 - y^2 = 1$	$X^2 - Y^2 - Z^2 = 0$

Notice that the last four equations are all homogeneous of degree two. Also, the projective equation of the parabola and the first hyperbola are the same with a permutation of the variable. So that gives you a hint that the last four plane figures in fact all come from similar kinds of objects. In general, the formula of a conic section in the projective plane is of the form

$$a \cdot X^2 + 2b \cdot XY + 2d \cdot XZ + c \cdot Y^2 + 2e \cdot YZ + f \cdot Z^2 = 0. \tag{13}$$

In matrix notation, we can write it as

$$(X, Y, Z) \cdot \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix} \cdot \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = 0 \tag{14}$$

Let M be the matrix $\left(\begin{smallmatrix} a & b & d \\ b & c & e \\ d & e & f \end{smallmatrix}\right)$ and $q = (X, Y, Z)$ be the column vector representing a point. Then the equation above can be written in the following more compact form

$$q^T M q = 0, \tag{15}$$

where q^T is the transpose of q. Notice that M is symmetric, i.e. it is the same as its transpose. In fact, non-symmetric M' can be replaced with $M = M' + M'^T$ to make it symmetric without affecting our definition of conic later. If the determinant of M is 0, then set of points satisfying equation (15) will be union of lines. Thus, we want $\det(M) \neq 0$. In that case, we will define a conic with a symmetric 3×3 real matrix M to be

$$\mathcal{C}_M := \{q = [x : y : z] \in \mathbb{RP}^2 \mid q^T M q = 0\}$$

Remark: Notice that different matrices M can give rise to the same conic. For example, any nonzero scalar multiple of M give the same conic as M. Also, $A^T M A$ give the same conic if A is a nonzero 3×3 matrix. Using these notations, it is easy to classify conic sections as follows. At infinity, i.e. $Z = 0$, the equation becomes

$$aX^2 + bXY + cY^2 = 0.$$

Let $\Delta = b^2 - 4ac$ be the discriminant. Then if $\Delta > 0$, then this quadratic equation has two solutions and the conic section has two intersections with the line at infinity. In this case, we call the conics *hyperbola*. Similarly, if $\Delta = 0$, resp. $\Delta < 0$, we have a *parabola*, resp. *ellipse*.

4 Tangents and Dual Conics

In the case of parabola above, we see that the line at infinity is one of its tangent lines. With this description of parabola, it is easy to find its tangents at any other given points. To state the method, we need two lemmas at first.

Lemma 1. *Let M be a symmetric real 3×3 matrix and let ℓ be a line $\lambda a + \mu b$ given by two distinct points $a, b \in \mathbb{RP}^2$. Then the intersection of ℓ and $\mathcal{C}_M$ corresponds to the solutions (λ, μ) of the homogeneous system*

$$(\lambda, \mu) \cdot \begin{pmatrix} (a^T M a) & (b^T M a) \\ (b^T M a) & (b^T M b) \end{pmatrix} \cdot \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = 0 \tag{16}$$

In particular, there are at most two real solutions if the determinant of the matrix above does not vanish.

The proof of this lemma follows from expanding $(\lambda a + \mu b)^T M (\lambda a + \mu b) = 0$, which exactly describes the intersections between the line ℓ and the conic $\mathcal{C}_M$. So this formula gives us a way to find out the intersection between a line and a conic. The next lemma tells us that it cannot be the whole line itself.

Lemma 2. *A conic $\mathcal{C}_M$ with M symmetric and $\det(M) \neq 0$ cannot contain a projective line.*

The proof uses the fact that if a projective line satisfies the equation $q^T M q$, then the matrix M can be factored as

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \cdot \begin{pmatrix} u' & v' & w' \end{pmatrix} + \begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} \cdot \begin{pmatrix} u & v & w \end{pmatrix},$$

which has determinant 0. So $\mathcal{C}_M$ cannot contain a projective line. With these two lemmas, we can find the tangent line at point q to a conic $\mathcal{C}_m$ as below.

Theorem 3. *Let $\mathcal{C}_M$ be a conic with $\det(M) \neq 0$ and $q \in \mathcal{C}_M$. Then the homogeneous coordinate of the tangent line at q is given by $M \cdot q$.*

Proof. Since $q \in \mathcal{C}_M$, we have $q^T M q = 0$. Let $q' \in \mathbb{RP}^2$ be a point on the line determined by $M \cdot q$ different from q. So q' satisfies

$$(q')^T M q = 0.$$

If q' also lies on the conic $\mathcal{C}_M$, then $(q')^T M q' = 0$ and the 2×2 matrix in equation (16) is the zero matrix. That means $\mathcal{C}_M$ contains the line determined by q, q'. However, that is impossible by lemma 2. Thus, the line determined by $M \cdot q$ has only one intersection with the conic $\mathcal{C}_M$, and is tangent to $\mathcal{C}_M$. □

Now we know how to find the tangent lines at a given point on a conic $\mathcal{C}_M$, we can consider the collection of tangent lines to $\mathcal{C}_M$. Using the duality between lines and points in $\mathbb{RP}^2$, this set can be transformed into another conic $\mathcal{C}_M^*$, called the *dual conic* of $\mathcal{C}_M$. With the fact that $M^T = M$, we have

$$0 = q^T M q = q^T M^T M^{-1} M q = (Mq)^T M^{-1} (Mq).$$

By the theorem above, we see that the conic determined by M^{-1} is the same as $\mathcal{C}_M^*$, i.e. $\mathcal{C}_{M^{-1}} = \mathcal{C}_M^*$. To represent $\mathcal{C}_M^*$, we will use $\det(M) \cdot M^{-1}$ instead (*remember they give rise to the same conic*). Now the problem of finding common tangents to two given parabolas becomes the problem of finding the intersections of their dual conics. By Bézout's Theorem, two conics can have at most 4 intersections. So there are at most 4 common tangents to two parabolas. Since any parabola has the line at infinity as their tangent, there are at most 3 common tangents in the affine plane $\mathbb{R}^2$. As we saw last week in certain cases, they corresponds to solutions of a cubic equation.

Finally, let's put everything together and find the common tangent of the following two parabolas in $\mathbb{R}^2$ using homogeneous coordinate

$$y = \frac{x^2}{2}, \quad y^2 = -4x.$$

References

[1] H.S.M. Coxeter, *The Real Projective Plane*, McGraw-Hill 1992

[2] J. Richter-Gebert, *Perspectives on Projective Geometry*, Springer 2011