Double summations and Root of Unity Filters

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Summations are prevalent across mathematics, scientific fields, and industry. In this handout, I cover a few tricks to evaluate sums. This is only scratching the surface of summation methods The first main topic, "changing the order of summation," is arguably a method that is ubiquitous in math and outside. The second main topic is the "Root of unity filter," something that I plan to use later this quarter.

Credit to Evan Chen for ideas. I recommend Evan's handout for further summation methods for olympiads. `https://web.evanchen.cc/handouts/Summation/Summation.pdf`

1 Changing the order of summation

Broadly speaking, a summation looks like the following:

$$\sum_{n \in S} f(n),$$

where S is the set over which n ranges, and we apply the function f to get the summands. A double summation looks like the following:

$$\sum_{m \in S_1} \sum_{n \in S_2} f(m,n).$$

When the sets S_1, S_2 are finite, then we can state the following, known as **changing the order of summation**:

$$\sum_{m \in S_1} \sum_{n \in S_2} f(m,n) = \sum_{n \in S_2} \sum_{m \in S_1} f(m,n).$$

I will state this without proof, but it is just arguments of commutativity and associativity. When the sets are not finite, you have to consider convergence issues. If the whole sum is absolutely convergent, then you can change the order. Double infinite sums are outside the scope of this worksheet.

Exercise 1. (Average fixed points of Permutations) Let n be a positive integer. Represent permutations on n elements as functions $[1,\dots,n] \to [1,\dots,n]$. The number of fixed points of a permutation σ is defined as the number of integers $i \in [1,\dots,n]$ such that $\sigma(i) = i$. Then the average number of fixed points of σ over all $n!$ permutations equals 1.

Remark: I claim that it is very inconvenient to try to brute force the problem in this way: compute the number of permutations that have exactly 0 fixed points, then the number of permutations that have exactly 1 fixed point, etc. The expressions for these quantities are quite inconvenient.

Proof: Define $F(\sigma)$ to be the number of fixed points of σ. Define $I(\sigma, i)$ to be the indicator function which is 1 if $\sigma(i) = i$ and 0 otherwise. Then

$$F(\sigma) = \sum_{i=1}^{n} I(\sigma, i).$$

The average number of fixed points equals the following

$$(n!)^{-1}\sum_{\sigma}F(\sigma)=(n!)^{-1}\sum_{\sigma}\sum_{i=1}^{n}I(\sigma,i)=(n!)^{-1}\sum_{i=1}^{n}\sum_{\sigma}I(\sigma,i)$$

For any i, $\sum_{\sigma} I(\sigma,i)$ equals the number of permutations that fix i, which equals $(n-1)!$ Then the desired quantity equals

$$(n!)^{-1}\sum_{i=1}^{n}(n-1)!=(n!)^{-1}\cdot n(n-1)!=1.$$

Changing the order of summation is also implicated in the classical result of **Linearity of Expectation**. I have left a limited, discrete version of linearity of expectation as an exercise (in order to avoid technicalities with continuous distributions which are out of scope). Also, this previous result can be stated in terms of linearity of expectation.

Exercise 2. Let $G(V,E)$ be a finite graph. A subset T of V is called **dominating** if each vertex v is in T and/or adjacent to an element of T. Prove that the number of dominating subsets is odd.

Note: This is equivalent to USA TST 2010 Number 6, solved by very few people at MOP that year. However the difficulty is primarily in the difficulty of the argument. The result was already known by certain specialists. `https://www.win.tue.nl/~aeb/preprints/domin4a.pdf`

Proof: We will consider ordered pairs of subsets of V. A pair (T_1,T_2) is called **good** if the subsets are disjoint, and moreover, for any $v_1\in T_1$ and $v_2\in T_2$, v_1 and v_2 are not adjacent. T_1 and T_1 are allowed to be empty. Let $I(T_1,T_2)$ be the indicator function for whether the pair is good. Let $P(V)$ be the powerset of V. Consider the following double sum mod 2.

$$\sum_{T_1\in P(V)}\sum_{T_2\in P(V)}I(T_1,T_2)$$

As an integer, this sum equals the number of good pairs.

Claim: For any T_1 which is not a dominating subset, the inner sum evaluates to a power of 2 greater than 1. For any T_1 which is a dominating subset, the inner sum evaluates to exactly 1, which is when T_2 is empty. Thus, mod 2, the sum evaluates to the number of dominating subsets mod 2. This seems like barely any information at all!

Next, we observe that for any good pair (T_1,T_2), the pair (T_2,T_1) is good. Each good pair is swapped to a different good pair except (ϕ,ϕ), the pair of empty sets (since (T,T) is good iff T is the empty set). The number of good pairs is odd.

We know that the number of good pairs equals the number of dominating subsets mod 2. This completes the proof

2 Root of Unity Filter

A root of unity filter is a trick that generally evaluates the sum of a periodic selection of terms from another "complete sum." This trick is related to the field of Fourier analysis, which is a rigorous study of sine waves (or cosine, exponential), constructing periodic signals, and much more. Fourier analysis is not needed for this topic.

A classic problem is the following. We wish to compute

$$\sum_{j=0,2|j}^{1000}\binom{1000}{j}.$$

In order to do this, we observe

$$(1+1)^{1000} = \sum_{j=0}^{1000} \binom{1000}{j}$$
$$(1-1)^{1000} = \sum_{j=0}^{1000} \binom{1000}{j}(-1)^j.$$

We add these two equations and collect like terms of $\binom{1000}{j}$. A key fact is that $1+(-1)^j$ equals 2 if j is even and 0 otherwise. We get the following:

$$2^{1000} + 0 = \sum_{j=0, jeven}^{1000} 2 \cdot \binom{1000}{j}.$$

$$\sum_{j=0,2|j}^{1000} \binom{1000}{j} = 2^{999}.$$

How about computing the following?

$$\sum_{j=0,2|j}^{1000} \binom{1000}{j}.$$

Hint: Let ω be a third root of unity, so $\omega^3 = 1$ and $\omega! = 1$. Expand $(1+\omega)^{1000}$ with the binomial theorem, and do th e same for $\left(1+\omega^2\right)^{1000}$.

3 Exercises

Exercise 3. Let n be a positive integer. Prove that $\sum_{d|n,d>0} \varphi(d) = n$. Write this argument in terms of changing the order of summation as follows. For $d \mid n, d > 0$, prove that $\varphi(d) = \sum_{i=1}^{n} I(gcd(i,n), d)$, where I is an indicator function that evaluates to 1 if the arguments are equal and evaluates to 0 otherwise.

Exercise 4. Prove linearity of expectation in the following limited setting. N is a positive integer. X, Y are random variables with values in $[1, 2, \ldots, N]$ and not necessarily independent. Prove that $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$.

Recall that $\mathbb{E}(X) = \sum_{m=1}^{n} mp(X=m)$, etc. The usual proof involves tricky rewriting of double sums and writing one summation as the sum of two summations.

Exercise 5. Restate the argument, using linearity of expectation, that the average of fixed points of a permutation on n elements equals 1 in a by letting σ be a random variable that equals each of the $n!$ permutations with equal probability. (The argument looks quite similar.)

Exercise 6. Denote the number of positive factors of a positive integer n to be $f(n)$. Compute

$$\sum_{i=1}^{100} f(i).$$

Use "changing the order of summation" to equate this to an easier sum. See if you can evaluate the sum by adding only 10 numbers.

Exercise 7. (2019 US Ersatz MO #4 with construction given) Let p be a prime and $n = p(p-1)$. Compute the following

$$1^n + 2^{n-1} + 3^{n-2} + \cdots + n^1 \pmod{p}.$$

Exercise 8. (AMSP 2011 NT3 Exam) Let n be a positive integer. Prove that

$$\sum_{k=1}^{n} \varphi(k) \lfloor n/k \rfloor = \frac{n(n+1)}{2}.$$

Exercise 9. How many subsets of $\{1,2,\dots,1000\}$ have sum divisible by 3? (This uses generating functions in one variable and a root of unity filter. Consider $(1+x)(1+x^2)(1+x^3)\cdots$)

Exercise 10. Let k be a positive integer, $p = 6k+1$, and assume that p is prime. Let g be a primitive root mod p. Let $h = g^6$. Evaluate the following, mod p. The result should be a definite sum (no summation notation).

$$\sum_{i=0}^{k-1} \left(1 + h^i\right)^{3k}.$$

Note: This summation was used in the 2010 USA TST Problem 9. The problem's full solution is not much longer and uses that $x^{3k} \in -1, 0, +1 \bmod p$. However, thinking of this intermediate summation required a lot of insight.