UCLA Math Circle: Origami and Construction II

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0. Review from Last Time

Last time, we learned how to trisect an arbitrary angle using a combination of the following folds

(F1.) Given two points, one can make a fold to identify them, thus creating their perpendicular bisector.

(F2.) Given a line ℓ and a point P, one can create the the reflection of P across ℓ.

(F3.) Given a line ℓ and a point P, one can fold a line perpendicular (or parallel) to ℓ and passes through P.

(F4.) Given an arbitrary angle, one can fold its angle bisector

(F5.) Given a line ℓ and points P, Q, then whenever possible, one can fold a line passing through Q and reflect P onto ℓ.

(F6.) Given two lines ℓ, m and points P, Q, then whenever possible, one can make a fold such that P is reflected onto ℓ and Q is reflected onto m.

We also quantified the strength of the folds $(F1) - (F4)$ by studying the set of points $\mathcal{P}$ and set of lines $\mathcal{L}$ in the plane obeying the following axioms:

(A0) There are at least 3 non-colinear points in $\mathcal{P}$.

(A1) If $P, Q \in \mathcal{P}$, then $\ell_{PQ} \in \mathcal{L}$.

(A2) Given non-parallel lines $\ell, m \in \mathcal{L}$, their intersection is in $\mathcal{P}$.

(A3) Given two points $P, Q \in \mathcal{P}$, the perpendicular bisector of PQ is in $\mathcal{L}$.

(A4) For any angle formed by two lines $\ell, m \in \mathcal{L}$, its angle bisector is also in $\mathcal{L}$.

Last time, we showed that $(F4)$ cannot be accomplished under $(A0) - (A3)$. So we added $(A4)$ to the list of axioms. Similarly, one could show that $(F5)$ and $(F6)$ cannot be accomplished under $(A0) - (A4)$. So we will let $\mathcal{P}$ and $\mathcal{L}$ satisfy the following additional axioms:

(A5) Given $\ell \in \mathcal{L}$ and $P, Q \in \mathcal{P}$. If there is a line m such that it passes through Q and reflects P onto ℓ, then $m \in \mathcal{L}$.

(A6) Given $\ell_1, \ell_2 \in \mathcal{L}$ and $P_1, P_2 \in \mathcal{P}$. If there is a line m such that it reflects P_i onto ℓ_i for $i = 1, 2$ simultaneously, then $m \in \mathcal{L}$.

To understand the kind of line produced by $(A5)$ and $(A6)$, let us first quickly review the definition of properties of parabola.

1. Parabola and $(A5)$

Definition 1. *Given a line d and a point F not on d, the set of points that are equidistance away from the P and d is called a parabola. The point F is called the* ***focus*** *and line d is called the* ***directrix****.*

The plane is divided by the parabola into two regions. We will call the region that contains the focus the *interior* of the parabola. The other region will be called the *exterior* of the parabola.
For $(A5)$, if we fix the point P, line ℓ and vary the point Q, then the collection of lines form the shape of a parabola. So it is not too difficult to see that $(A5)$ constructs the line going through Q and tangent to the parabola with focus P and directrix ℓ. From this, one could tell that the line described in $(A5)$ exists exactly when point Q is not in the interior of the parabola.
Exercise: Let $P = (0, a)$ be the coordinate of the focus and $y = -a$ be the directrix. Show that the equation of the parabola is

$$y = \frac{x^2}{4a}.$$

Now let $P = \left(x_0, \frac{x_0^2}{4a}\right)$ be a point on this parabola. Show that the slope of the tangent line at P is $\frac{x_0}{2a}$ without using calculus.
Using $(A5)$, one can take square root of any positive real number r in the following procedure. Let $P = (0, 1), Q = (0, -\frac{r}{4})$ and $\ell : y = -1$. Then P and ℓ determines the parabola $y = \frac{x^2}{4}$. By $(A5)$, we can construct the line tangent to $y = \frac{x^2}{4}$ and going through Q. It can be easily checked from the exercise that the point of tangency is $(\sqrt{r}, \frac{r}{4})$. So the x-coordinate $\sqrt{r}$ is a constructible length. In fact, any straightedge and compass construction can be accomplished using $(A1)$ – $(A5)$.

2. Parabolas and $(A6)$

Similarly in $(A6)$, let $\mathbb{P}_i$ denote the parabola with focus P_i and directrix ℓ_i for $i = 1, 2$. Then the line described in $(A6)$ is simultaneously tangent to both $\mathbb{P}_1$ and $\mathbb{P}_2$. Here the existence and uniqueness of such a line is less clear than in the case of $(A5)$. If you have learned some projective geometry, then the following remark roughly sketches out a reason for the existence of common tangent lines. Otherwise, you could take the existence for granted in the cases that we will look at.
Remark: It is much easier to consider parabolas in the projective plane. One of the reasons is the duality between points and lines in the projective plane. Using this, one can consider the dual parabolas, whose points parametrize the tangent lines of the original parabolas. Thus, find common tangent becomes the problem of finding intersection between the dual parabolas. Since both of these are defined by a degree 2 homogeneous polynomial, Bézout's theorem tells us that

there are $2 \cdot 2 = 4$ intersection points, counting multiplicity. That means there are at most 4 tangent lines. In fact, the line at infinity is always tangent to a parabola in the projective plane. So there are at most 3 common tangent lines in $\mathbb{R}^2$. For example, the parabolas represented by the equations $y = x^2$ and $x = y^2 + 4$ have three distinct common tangents, whereas the parabolas represented by the equations $y = x^2$ and $y = 2x^2 + 1$ have no common tangents in $\mathbb{R}^2$.
In $(A6)$, the fold that creates the common tangent is called the *Beloch fold*, after an Italian mathematician named Margharita Piazolla Belloch. For more information, see [1]. Using this fold,, we could find solutions to cubic equations of the form

$$X^3 + aX + b = 0. \tag{1}$$

Any cubic equation can be changed into the form (1) by a linear change of variable in X. The following series of exercises will show that the slope of a common tangent of two particular parabolas is a solution of the equation above.

Exercises

1. Find the equations of the parabolas $\mathbb{P}_0, \mathbb{P}_1$ with these focus and directrix

1. $P_0 = (0, \frac{1}{8}), \ell_0 : y = -\frac{1}{8})$
2. $P_1 = (\frac{b}{2}, \frac{a}{2}), \ell_1 : x = -\frac{b}{2}$.

2. Let $P_i = (x_i, y_i)$ be a point on the parabola $\mathbb{P}_i$ and μ_i the slope of the tangent line to $\mathbb{P}_i$ at P_i. Show that

$$\mu_1 = x_1,\ y_1 = \frac{\mu_1^2}{2},\ \mu_2 = \frac{b}{y_2 - \frac{a}{2}},\ x_2 = \frac{b}{2\mu_2^2}. \tag{2}$$

3. Let μ be the slope of the line connecting points P_1 and P_2. Suppose that $\mu_1 = \mu_2 = \mu$. Show that μ satisfies the equation

$$\mu^3 + a\mu + b = 0. \tag{3}$$

3. Beloch Square and Construction of $\sqrt[3]{2}$

The Beloch square is defined as follows.

Definition 2. *Given two points A, B and two lines r, s, we can construct a square $WXYZ$ such that two adjacent corners X, Y lie on r, s respectively, and the sides WX and YZ, or their extensions, pass through A and B respectively.*

Exercises:

1. Show that the Beloch square can be constructed using the Beloch fold.

2. Let $A = (-1, 0), B = (0, -2), r : x = 0, s : y = 0$. Denote X the corner of the Beloch square constructed from A, B, r, s. Show that X has coordinate $(0, \sqrt[3]{2}$.

Besides using the Beloch square, here is another procedure to construct $\sqrt[3]{2}$, which is more convenient to fold, but less clear to prove.

Procedure to Construct $\sqrt[3]{2}$

Step 1. Take a piece of square paper, and fold two parallel lines that divide the square into three equal rectangles.

Step 2. Orient the square so that the folds are horizontal. Denote the lower parallel segment by ℓ_1 and higher parallel segment by ℓ_2. Denote the vertices of the square by A, B, C, D from the top left corner in the clockwise direction. Let E be the point on BC and ℓ_1. So $|BE| = 2|CE|$.

Step 3. Make a fold such that point C is folded onto segment AD and point E is folded onto ℓ_2. Denote the image of C on AD under the fold by C'.

Exercise: Show that the ratio between $|AC'|$ and $|C'D|$ is $\sqrt[3]{2}$.

References

[1] T. Hull, *Solving Cubics With Creases: The Work of Beloch and Lill*, Amer. Math. Monthly, April 2011, 307-315