Introduction to Homotopy Theory

Adapted from a worksheet by Sanath Devalapurkar

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Homotopy theory is a very interesting branch of a subfield of mathematics called algebraic topology. Today, I'll give an introduction to a basic notion in homotopy theory, namely the notion of homotopy groups. The hope is that you develop an intuition for these objects, so don't think too rigorously!

1 Homotopy

Topology is the study of spaces and continuous functions between them. In this worksheet, our spaces will be subsets of Euclidean space $\mathbb{R}^n$ (for various values of n). Let X and Y be two subsets of Euclidean spaces (perhaps of different dimensions. As you may recall from earlier worksheets, these are metric spaces, so we have a definition of what it means for a function $f : X \to Y$ to be continuous. We can use continuous functions to make some pictorial ideas more rigorous:

Problem 1.1. If a, b are two points on X, how can we define a *path* from a to b in X?

This week, we'll also try to use continuous functions to understand what it means for two spaces to be equivalent.

We say that two spaces are *homotopy equivalent* if we can bend, stretch, etc., and *squish* them to make them "equivalent". For instance, it is possible to take the unit disk in $\mathbb{R}^2$, or even the entire space $\mathbb{R}^n$, and squish it down to just a single point. (Can you see why?) A space that is homotopy equivalent to a point is called *contractible*.

Problem 1.2. A common joke in mathematics is that a topologist is someone who can't distinguish between a donut and a mug. Can you see why? (Think of homotopy equivalences.)[1]

Problem 1.3. Can you find five letters (capital/uppercase) which are contractible? (Think of a letter as a subset of the plane on which it's drawn.)

[1] Technically, these spaces are homeomorphic, which is a stronger notion than homotopy equivalence. Finding an example of two things which are homotopy equivalent but not homeomorphic is Exercise 1.4

Problem 1.4. Give an example of two uppercase letters which can*not* be deformed into one another *without* squishing, but *can* with squishing.

Can we be more (somewhat) precise about homotopy equivalences? Let X and Y be topological spaces, and let $f_0 : X \to Y$ and $f_1 : X \to Y$ be continuous maps of topological spaces. Say that f_0 and f_1 are *homotopic* if for each real number t between 0 and 1 (inclusive), there are maps $f_t : X \to Y$ such that you can "infinitesimally deform" f_0 to each f_1 by shifting f_0 along the collection of the specified maps f_t. This simply means that we can "deform" f_0 into f_1, continuously, or without jumping.

Problem 1.5.

- Let $f, g : S^1 \to S^1$ be the following two functions from the circle to itself: let f be the identity, and let g send everything to a fixed point $x \in S^1$. Explain why f and g are not homotopic.
- Let $f, g : S^1 \to S^2$ be the following two functions from the circle to the sphere: let f map the circle to the equator of the sphere, and let g send everything to a fixed point $x \in S^2$. Explain why f and g are homotopic.

Definition 1.1. Let X, Y be topological spaces. If there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity maps on X and Y, respectively, then we say that X is homotopy equivalent to Y.

Two homotopy equivalent spaces have the "same number of holes". Let's try finding some examples.

Problem 1.6. Verify that the unit disk $\mathbb{D}^2 = \{(x, y) : x^2 + y^2 \leq 1\}$ is contractible with an explicit homotopy equivalence.

Problem 1.7. Find a homotopy equivalence between the cylinder, which is a "product" $S^1 \times [0, 1]$, and the circle S^1.

The next exercise will ask you to *explicitly* write down a homotopy equivalence.

Problem 1.8. Let X be the unit circle S^1 in the plane $\mathbb{R}^2$, and let Y be the plane with the origin removed. Define an *explicit* homotopy equivalence between S^1 and $\mathbb{R}^2 - \{(0, 0)\}$. Hint: Use the "inclusion" $S^1 \to \mathbb{R}^2 - \{(0, 0)\}$ and Pythagoras' theorem - what's a good way to assign, to a pair of nonzero real numbers, a point on the unit circle? What this requires is for you to specify two maps $f : S^1 \to \mathbb{R}^2 - \{(0, 0)\}$ and $g : \mathbb{R}^2 - \{(0, 0)\} \to S^1$.

2 The Fundamental Group

Now, suppose $X = [0, 1]$, and let Y be a topological space. A *path* in Y is a continuous function $f : [0, 1] \to Y$; so homotopies between maps of topological spaces give homotopies between paths. A *loop* in Y is a path $f : [0,1] \to Y$ such that $f(0) = f(1)$; this point $x = f(0)$ is called the *basepoint* of the loop.

If you've got a collection of loops on a topological space, you can consider the set of loops *modulo homotopy*, which simply means that you identify two loops together if they're homotopic.

Let X be a topological space. Choose a point $x \in X$, and consider all the loops whose basepoint is x modulo homotopy. This is called the *fundamental group* of X based at x, and is denoted $\pi_1(X, x)$. If X is *path-connected*, meaning that for any two points $a, b \in X$, there is a path from a to b, then the choice of basepoint doesn't matter. Why is this true?

Now, why not call it the fundamental *set* of X based at x? To make it a *group*, we need to define some kind of multiplication operation.

Definition 2.1. Suppose γ is a path from a to b and Γ is a path from b to c. Define the "product" of γ and Γ, denoted as $\gamma * \Gamma$, to be the path from a to c, defined by $(\gamma * \Gamma)(t) = \gamma(2t)$ for $0 \leq t \leq \frac{1}{2}$, and $(\gamma * \Gamma)(t) = \Gamma(2t-1)$ for $\frac{1}{2} \leq t \leq 1$.

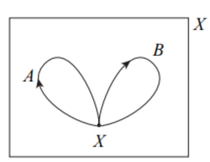
Problem 2.1. Find two paths on the unit circle S^1, γ_1 and γ_2, which are not loops, such that $\gamma_1 * \gamma_2$ is a loop that goes around the circle.

This multiplication operation turns out to satisfy these nice properties:

1. "Multiplication" of two loops (see above) is itself a loop.
2. The multiplication is associative (up to homotopy).
3. There is an identity loop such that when composed with any other loop it yields the loop itself.
4. For every loop there is an inverse loop such that when the two are composed they yield the identity.

Problem 2.2. We say that multiplication is associative up to homotopy. To prove this, let g_1, g_2, g_3 be paths from a to b to c to d. Show that $(g1 * g2) * g3$ is homotopic to $g1 * (g2 * g3)$.

Problem 2.3. Draw out some pictures to convince yourself that the rest of these rules are satisfied. (This visualization of the "multiplication" in $\pi_1(X, x)$ may be helpful (the image has been taken from the Princeton Companion to Mathematics):



Problem 2.4. Fundamental groups are not always commutative. Explain why in the picture above, $A * B$ is not the same as $B * A$.

If X and Y are homotopy equivalent, then $\pi_1(X)$ is isomorphic to $\pi_1(Y)$; in other words, there is a map $\phi : \pi_1(X) \to \pi_1(Y)$ which is a bijection which takes the multiplication $\phi(f \circ g) \to \phi(f) \circ \phi(g)$. Let's see this with the example we established below.

Problem 2.5. Recall that $\mathbb{R}^2 - \{(0,0)\}$ is homotopy equivalent to the circle. Compute the fundamental groups of $\mathbb{R}^2 - \{(0,0)\}$ and S^1. (These are path-connected (why?), so forget about basepoint issues.)

Hint 1: this is actually not that hard; just draw stuff out!

Hint 2: Because $\mathbb{R}^2 - \{(0,0)\}$ is homotopy equivalent to the circle, it suffices to only compute fundamental group of one of the two spaces. It's easier to compute π_1 of S^1.

In topology, $S^1 \times S^1$ called a torus; why? A torus is simply a donut. Let's imagine a circle; choose a point on this circle. Then draw another circle, "perpendicular" to the first circle, around this point. If you push the new circle around the old circle, until you reach the starting point, you end up with a donut.

To visualize this, check out the following youtube video: `https://www.youtube.com/watch?v=nLcr-DWVEto`

Problem 2.6. Let T^2 denote the torus $S^1 \times S^1$. What is $\pi_1(T^2)$? More generally, if T^n denotes the n-torus $\underbrace{S^1 \times \cdots \times S^1}_{n \text{ times}}$, what do you think $\pi_1(T^n)$ is?

Problem 2.7. Show that T^n is *not* homotopy equivalent to T^{n+1}.

Problem 2.8. Convince yourself that $\pi_1(X, x)$ is the collection of basepoint-preserving maps from the circle $S^1 \to X$.

Recall that we had said earlier that "Two homotopy equivalent spaces have the 'same number of holes"'. How do we make this a bit more precise?

Let $S^1 \vee S^1$ denote the wedge product of two circles, which, in other words, means we join the two circles at a single point. Then $\pi_1\left(S^1 \vee S^1\right)$ is the *free group on 2 generators* a, b. This is the collection of all "words" one can write using a, b, a^{-1}, b^{-1}, such that we can simply "cancel" $aa^{-1}, a^{-1}a, bb^{-1}$, and $b^{-1}b$.

Problem 2.9. Show that this is true using Exercise 2.5. Show also that $S^1 \vee S^1$ is homotopy equivalent to $\mathbb{R}^2$ with two points removed. More generally, convince yourself that π_1 of $\mathbb{R}^2$ with n points removed is isomorphic to the free group on n generators.

Problem 2.10. Let S^2 denote the 2-sphere; this is simply a the boundary of a ball. What is $\pi_1(S^2)$?