For our purposes, a geometric transformation is a map $T$ from the Euclidean plane to itself. It’s often helpful to envision such transformations as “moving” each point $A$ to its image point $T(A)$; we’ll later look at images of entire sets of points (such as lines or triangles).

- **Translations** (by a translation vector $\vec{w}$):

  \[
  In\ the\ complex\ plane:\ 
  z \mapsto z + w \]

- **Homotheties or dilations** (with respect to a center $O$, by a *dilation factor* of $\lambda$):

  \[
  In\ the\ complex\ plane\ (origin = O): \ 
  z \mapsto \lambda \cdot z \]

- **Rotations** (by a counterclockwise angle $\theta$ rad, with respect to a *center of rotation* $O$):

  \[
  In\ the\ complex\ plane\ (origin = O): 
  z \mapsto z \cdot e^{i\theta} = z(\cos \theta + i \sin \theta) \]
Reflections (with respect to a line $\ell$):

In the complex plane (real axis = $\ell$):

$$z \mapsto \overline{z} = \text{Re}(z) - i\text{Im}(z)$$

[Recap: Absolute values, arguments and conjugation.]

More generally, if $\lambda \neq 0$ and $z_0$ are complex numbers, then the map

$$z \mapsto \lambda(z - z_0) + z_0 = \lambda z + z_0(1 - \lambda)$$

is a rotational homothety (a rotation plus a homothety with the same center) centered at $z_0$, with dilation factor $|\lambda|$ and angle $\arg \lambda$. Hence if $w$ is a complex number,

$$z \mapsto \lambda z + w \quad \text{is} \quad \begin{cases} \text{a translation by } w, & \text{if } \lambda = 1, \\ \text{a rotational homothety centered at } \frac{w}{1-\lambda}, & \text{if } \lambda \neq 1. \end{cases}$$

Similarly, the reflection with respect to a line at angle $\theta$ with the real axis can be represented as

$$z \mapsto e^{2i\theta} \overline{z} + w,$$

where $w$ is the reflection of the origin, (1) so all compositions of the transformations that we are concerned with today can be represented as $z \mapsto \lambda z + w$ or as $z \mapsto \lambda \overline{z} + w$ for some $\lambda$ and $w$.

**Problem 1.** (Preserving properties) Prove the following using complex numbers:

(a) A nontrivial rotational homothety preserves exactly one point (which one is it?). Does a translation preserve any point (i.e., keep it fixed)? How about reflections?

(b*) Translations, rotations, homotheties and reflections map lines to lines, and circles to circles. 

*Hint:* a line consists of points $\{z = at + b : t \in \mathbb{R}\}$ for some $a$ and $b$, while a circle consists of points $\{z : |z - z_0| = r\}$, for some $z_0$ and $r$.

(c*) Translations, rotations and homotheties preserve oriented angles, and reflections preserve unoriented angles.

(d*) Translations, rotations and reflections preserve lengths of segments, while homotheties dilate lengths by the dilation factor $|\lambda|$. How would you guess that homotheties affect areas?

**Problem 2.** (Tangent circles) Two circles are externally tangent at $P$. Let $AB$ be a line through $P$ such that $A$ belongs to the first circle and $B$ to the second one (none of them being $P$). Show that the tangent at $A$ to the first circle and the tangent at $B$ to the second circle are parallel.

*Hint:* First, reflect the figure with respect to the line of the centers; what happens to the circles and why? Next, consider a homothety with negative factor centered at $P$; use that there’s a unique circle passing through three points.
**Problem 3.** (Compositions) Prove the following using complex numbers:

(a) Two rotational homotheties with the same center **commute**, i.e. the order in which we perform them does not matter.

(b) The composition of a rotational homothety and a translation is a rotational homothety.

(c) The composition of any two reflections is either a rotation or a translation. When is it a translation?

**Problem 4.** (Two homotheties) Let $ABCD$ be a convex quadrilateral, and let $X$ be the intersection of its diagonals. Let $E \in DB$ such that $AE \parallel CD$, and let $F \in AC$ such that $DF \parallel AB$. Prove that $EF \parallel BC$.

**Problem 5.** (Two reflections) Let $ABCD$ be a square, and let $X$ and $Y$ be points on sides $CD$ and $BC$ respectively, such that $\angle XAY = \angle DAX + \angle YAB$. Let $C''$ be the reflection of $C$ with respect to $AX$, and let $C'''$ be the reflection of $C''$ with respect to $AY$. Show that $B$ is the midpoint of $CC''$.

*Hint:* The composition of these two reflections cannot be a translation (why?), so it must be a rotation (with what center?). Look at what $D$ maps to under this rotation to determine the rotation’s angle.

**Problem 6.** (Similar triangles)

(a) Let $\{A, B\}$ and $\{A', B'\}$ be pairs of distinct points in the plane. Using complex numbers, show that there is a unique rotational homothety or translation mapping $A \mapsto A'$ and $B \mapsto B'$.

(b) Let $\triangle ABC$ and $\triangle A'B'C'$ be similar triangles oriented alike. Show that there is a unique rotational homothety or translation mapping $A \mapsto A'$, $B \mapsto B'$, $C \mapsto C'$ (and thus $\triangle ABC \mapsto \triangle A'B'C'$). *Hint:* Use part (a) and Problem 1.

**Problem 7.** (Centroid) Let $\triangle ABC$ be a triangle, and let $A', B', C'$ be the midpoints of $BC, CA$ and $AB$ respectively. Note that $\triangle ABC$ and $\triangle A'B'C'$ are similar and oriented alike, and consider the corresponding rotational homothety mapping $A \mapsto A'$, $B \mapsto B'$, $C \mapsto C'$. What must its angle be, given that $BC \parallel B'C'$? What about its dilation factor? What does this imply about its center?

**Problem 8.** (Euler’s circle and line) Let $\triangle ABC$ be an acute, scalene triangle, and let $H$ be its orthocenter (i.e. the intersection of all three heights).

(a) Show that the reflection of $H$ across the line $BC$, as well as the reflection of $H$ across the midpoint of $BC$ are both on the circumcircle $(ABC)$.

(b) Let $M_A, M_B, M_C$ be the midpoints of the segments $BC, CA, AB$ respectively. Let $H_A, H_B, H_C$ be the feet of the heights from $A, B$ and $C$ respectively (so that $AH_A, BH_B, CH_C$ intersect at $H$). Finally, let $N_A$ be the midpoint of $AH$, and similarly $N_B$ and $N_C$. Show that $M_A, M_B, M_C, H_A, H_B, H_C, N_A, N_B, N_C$ all lie on a circle, which is called Euler’s nine-point circle. *Hint:* Perform a positive homothety centered at $H$ with dilation factor $\lambda = 1/2$, and show that the circumcircle $(ABC)$ is mapped to this Nine-point circle.

(c) By part (b) of this exercise, we have shown that $(ABC)$ is mapped to $(M_A M_B M_C)$ by a homothety centered at $H$. On the other hand, in problem 7, we have seen that $\triangle ABC$ is mapped to $\triangle M_A M_B M_C$ through a homothety centered at $G$, the centroid. Thus, there exist homotheties centered at both $G$ and $H$ that take $(ABC)$ into the nine-point circle. Use this to show that the centers of both $(ABC)$ and the nine-point circle lie on the line $HG$. This is called Euler’s line.
Problem 9. (Toricelli’s point). Let $\triangle ABC$ be an acute triangle (or, if you want to be precise, with angles less than $120^\circ$.) Find a way to construct the point $T$ inside $\triangle ABC$ which minimizes the sum $TA + TB + TC$ (assume such a point indeed exists). Hint: Perform a $60^\circ$ rotation centered at $B$, sending $A \mapsto A'$ and $T \mapsto T'$, and show that $TA + TB + TC = A'T' + T'T + TC$. Then use the triangle inequality.

Homework 1

Problem 1. (Rotation in Cartesian coordinates) Consider a counter-clockwise rotation centered at the origin $(0,0)$, of angle $\theta$. Show that a point of coordinates $(x, y)$ goes to the point $(x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta)$. Then, plug in $\theta = 90^\circ$, and see what you get. The angle $45^\circ$ is also interesting.

Hint: The point $(x, y)$ corresponds to the complex number $x + iy$, and the rotation by $\theta$ to multiplication by $e^{i\theta} = \cos \theta + i \sin \theta$.

Problem 2. Consider two distinct lines $\ell_1, \ell_2$ that intersect at a point $O$. Show that the reflections along $\ell_1$ and $\ell_2$ commute (i.e. the order in which you perform them does not matter) if and only if $\ell_1 \perp \ell_2$.

Hint: Use complex numbers and pick the origin (the complex number 0) to be at $O$. You can also pick the real line to be $\ell_1$ to make things easier, and let $\theta$ be the counterclockwise angle from $\ell_1$ to $\ell_2$. What do the reflections along $\ell_1$ and $\ell_2$ look like as complex functions (check out equation (1)), and what happens when you compose them in either order?