Sets and Venn Diagrams Part 1
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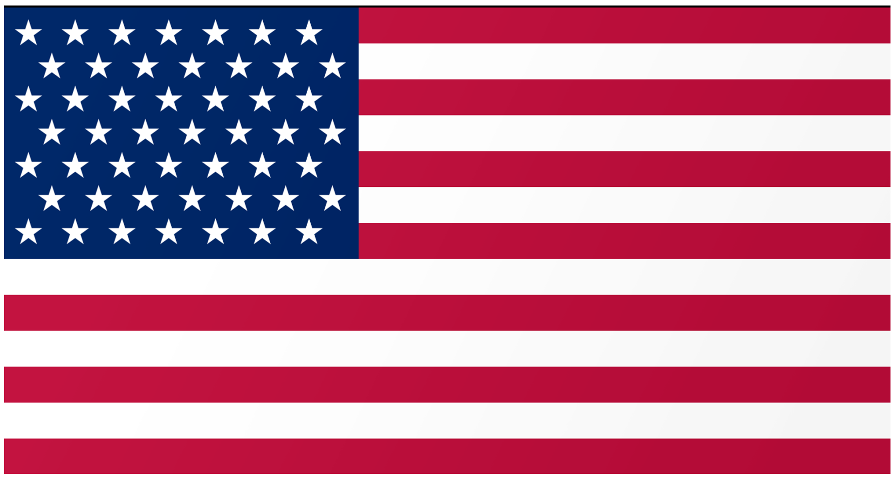
Warm-up:

What colors do we see on the American Flag?



Today, we're going to be talking about ***sets***. A ***set*** is a clearly defined collection of distinct objects. Note that this is not really a definition. To define means to explain in simpler terms. Instead, all we do is replace one word, a ***set***, by another, a ***collection***. Plus, the meaning of the words to clearly define is not clearly defined. The problem is that the notion of a set is as fundamental as it is deep. It is impossible to explain it in simpler terms. The best we can do at the moment is to show a bunch of examples.

Problem 1: Let's start by considering the factors of 6.

a. *What are these factors?*

   i. 1,2,3,6

b. We can introduce notation to represent them, ***a set***. This will be written with *curly braces*, { and }. In a set, the order doesn't matter. Let's write the set of factors of 6 and call it S.

$S = \{1,2,3,6\}$

c. Let's order these factors from smallest to largest. (Don't forget the curly braces)

$S_1=\{1,2,3,6\}$

d. We can introduce another set of notation, an ***ordered set or a list***. This is given by *round brackets*, (and), or *parentheses*.

i. *What do you think is the difference between a **set** and a **list**?*

The order matters for the list but not a set, so different orders of the same members of a set will produce different lists.

ii. Let's write the list of factors of 6 from least to greatest. (Don't forget the braces)

$L_1 = (1,2,3,6)$

e. Consider the two sets and two lists of letters:

$S_1= \{C, A, T\}$ $S_2= \{A, C, T\}$

$L_1= (C, A, T)$ $L_2 = (A, C, T)$

i. *Does $S_1 = S_2$? How do you know?*

Yes, each set contains the elements C, A, and T

ii. *Does $L_1 = L_2$? How do you know?*

No, while each set contains the elements C, A, and T, they are in a different order. As lists with a different order of elements are distinct, *L_1 is not equal to L_2*

f. *Returning to the warmup, if we let S_c be the set of colors of the American flag, what is in S_c ?*

S_c ={red, white, blue}

g. All the sets we've seen so far have something contained within it. What if a set does not contain any elements?

   i. *What do we call a set without any elements?*

      We call this the empty set, as it does not contain any elements.

   ii. We will represent the empty set as $\varnothing$.

   iii. {A set of pigs that can fly by themselves.} = $\varnothing$

   iv. Give your own description and example of the empty set.

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Notation:
$\in$: The fact that the number 6 is an element of the set S_1 is denoted as $6 \in S_1$. The fact the 7 is not an element of the set S_1 is denoted as $7 \notin S_1$.

$\subseteq$: For two sets S_1 and S_2, if every element of S_2 is also an element of S_1 ,then S_2 is a ***subset*** of S_1. In mathematical language, we write this as $S_2 \subseteq S_1$.

$\subset$: A subset of a set is called ***proper*** if it is not empty and is not equal to the original set.

Note: The notations $\subset$ and $\subseteq$ for sets are analogous to < and ≤ for numbers.

<u>Problem 2:</u> Some very special sets.

a. In the past, we've explored natural numbers. *What are natural numbers?*

   We defined natural numbers as the positive whole numbers or integers.
   Recall that 0 is not a natural number because it is a special number that was developed many tens or hundreds of thousands of years after people had learned to use natural numbers for counting

b. We can put all of the natural numbers into a set and give it a special symbol: N.

i. *Is 0 a part of the set of natural numbers? If not, how do you write it in mathematical language?*

$0 \notin N$

ii. *How would you formally read your answer for (i)?*

Zero does not belong to the set of natural numbers.

iii. Let's represent N in set notation:

$$N = \{1,2,3,...\}$$

c. We can also expand the set of natural numbers (N) to contain negative integers as well as zero. We will represent this set with the capital letter Z.

i. *Is 0 a part of the set of **integral numbers**, or just **integers**? How do you write it in mathematical language?*

$0 \in Z$

ii. *How would you formally read your answer for (i)?*

Zero belongs to the set of integral numbers (integers)

iii. Let's represent Z in set notation:

$$Z = \{.....-1,0,1,.....\}$$

d. We can further expand the set of ***integers***. Let $m \in Z$ and $n \in N$. The set Q of all the fractions m/n in the reduced form is called the set of rational numbers.

i. Is $\frac{2}{3} \in Q$? Yes, 2 and 3 are both integers

ii. What about $\frac{4}{5}$? Yes, 4 and 5 are both integers

iii. Is the number 0.55 rational? Why or why not?

Yes, we can represent it with the fraction 55/100=11/20.

e. *Can we find all of the elements of N in Z? How can we use notation to compare these two sets?*

$$N \subseteq Z$$

   i. *Is N a subset of N?* Yes, but not a proper subset

   ii. *Is Z a subset of Z?* Yes, but not a proper subset

   iii. *Is N a proper subset of Z?* If so, let's change our comparison to reflect this:

$$N \subset Z$$

f. Can we find all of the elements of Z in Q? How can we use notation to compare the two sets? *Is Z a proper subset of Q?*

$$Z \subset Q$$

Problem 3:

$S_1 = \{C, A, T\}$ $S_2 = \{A, C, T\}$

a. *Is S_1 a subset of S_2? Is S_1 a proper subset of S_2?*

   Yes, each element of S_1 can be found in S_2
   No, the sets contain the same elements

b. *Write down all the proper subsets of the set of colors of the US flag. Do they form a set? A list?*

   {red}, {white}, {blue}, {red, white}, {red, blue}, {blue, white}

   This forms a set, as we can write these in any order without the meaning changing. This is an artificial list, as there is not a significance in how we order the elements.

c. Consider the set S_3 = { cat, horse, lion, tiger, 0, 5, 100, dog}. Find the following subsets:

i. Subset of numbers = {0, 5, 100}

ii. Subset of animals = {cat, horse, lion, tiger, dog}

iii. Subset of animals whose names contain the letter l = {lion}

iv. Subset of common pets = {cat, dog}

v. Subset of colors = ∅

Red Hot Chilli Pepper

Move one digit to make the equality 101 - 102 = 1 correct

$101 - 10^2 = 1$

Problem 4: The number of elements in a set is called its ***cardinality***. In math language, we can write the cardinality of a set A as either |A| or card(A). We will use the former notation.

a. *Let U be the set of states in the United States. What is |U|?*

$|U|=50$

b. *What is |{0}|? Why?*

|{0}|=1, as there is is one element in the set, 0

c. *What is |∅|?*

|∅|=0, as this is the empty set and there are no elements in the set.

Next Time: Sets are a very important and useful idea. We can use this idea to mathematically explain a certain diagram we have previously explored: Venn Diagrams.

Challenge Questions

a. Decide which of the following two fractions is greater without cross-multiplying or bringing to the common denominator.

$$\frac{2017}{2018} < \frac{2018}{2019}$$

b. Given the sets below, find which sets are subsets of another (Use the notation we have learned so far):
   i. A = set of all flowers
   ii. B = set of all red objects
   iii. C = set of all tulips
   iv. D = set of all balloons
   v. E = set of all things you can use for birthday decorations

$A \subseteq A$, $B \subseteq B$, $C \subseteq C$, $D \subseteq D$, $E \subseteq E$

$C \subset B$, $C \subset A$, $D \subset E$

c. Consider the following sets:

A = {cat, horse, lion, tiger, 0, 5, 100, dog}

B = { 1 , 5, 100, panda, mouse, seahorse, lion}

   i. Can you make a new set that contains elements that are found both in A and B?

   C={lion, 5, 100}

ii. Can you make a new set that contains elements that are found in either A or B?

D={cat, horse, lion, tiger, 0, 5, 100, dog, 1, panda, mouse, seahorse}

d. How many subsets, including the empty set and the set proper, does a set A of the following cardinality have?

i. 2

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ie if A={1, 2}, we have sets {1,2}, {1}, {2}, and ∅

ii. 3

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ie if A={1, 2, 3}, we have sets {1, 2, 3}, {1, 2}, {1, 3}, {2, 3}, {1}, {2}, {3}, ∅

iii. 4

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ie if A={1, 2, 3, 4}, we have sets {1, 2, 3, 4}, {1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4}, {1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, {3, 4}, {1}, {2}, {3}, {4}, ∅

iv. 5

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ie if A={1, 2, 3, 4, 5}, we have sets {1, 2, 3, 4}, {1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4}, {1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, {3, 4}, {1}, {2}, {3}, {4}, and ∅, which all are found in the subsets for cardinality of 4, and we have sets {1, 2, 3, 4, 5}, {1, 2, 3, 5}, {1, 2, 4, 5}, {1, 3, 4, 5}, {2, 3, 4, 5}, {1, 2, 5}, {1, 3, 5}, {1, 4, 5}, {2, 3, 5}, {2, 4, 5}, {3, 4, 5}, {1, 5}, {2, 5}, {3, 5}, {4, 5}, and {5}

v. Do you see a pattern?

We see that there are 2^n subsets for a set with n elements, or a cardinality of n.

e. Given $|A| = n$, prove that the cardinality of the set of all the subsets of A is 2^n.

We can prove this by induction.

Induction base: We can start with n=1. Let's arbitrarily choose $A_1=\{1\}$. This gives us two subsets, the empty set, ∅, and {1}.

Assume that this is true for some some subset with n elements, such that $|A_n|=n$. This gives us that $|\{\varnothing, \{1\}, ... , \{n\}, \{1, 2\}, ... , \{1, 2, …, n\}\}|=2^n$

Consider the case for another set, A_{n+1}, with n+1 elements. For each subset of A_n, there are two subsets in A_{n+1}: the original subset and one containing the element n+1. Then, we have the set of subsets of A_{n+1} as $|\{\varnothing, \{n+1\}, \{1\}, \{1, n+1\}, ... , \{n\}, \{n+1\}, \{1, 2\}, \{1, 2, n+1\}, ... , \{1, 2, …, n\}, \{1, 2, …, n+1\}\}|$. We know that this is twice the number of subsets that we found for A_n, so the cardinality of set of all the subsets of A_{n+1} is $2*2^n$. From our lessons on exponents, we know that this is equal to 2^{n+1}, proving that the cardinality of the set of all the subsets of A is 2^n.

That is all that is needed for an induction proof. We have shown that the cardinality of the set of all the subsets of A_{n+1} is 2^{n+1} if the cardinality of the set of all the subsets of A_n is 2^n. As the cardinality of the set of all the subsets of A_1 is 2^1 the cardinality of the set of all of the subsets of A_2 must be 2^2. From this, we know it is true for n=3, which can be used to prove the case if n=4, and so forth.