A dive into Algebraic Topology

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These notes are supposed to be suggestive of what was covered during the class.
For the students: Please check the OneNote document as it contains the detailed notes.

0.1 Preliminaries

-A torus drawing

Example of equivalence relation and representative quotient: $\pmod n$

-Equivalence relation and representative quotient := Partition (Draw)

- Examples of groups: Rotations of a regular triangle, rotations of a square, rotations of n-gon, rotations of a circle, rotations of spheres along a fixed axis, affine transformations of plane. Other more abstract examples: $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}[x]$, etc.

Definition 1 (Groups). *A group is a set G together with a binary operation $* : GxG \to G$ so that*
1)$(g * h) * r = g * (h * r)$
2)There is a unique element $e \in G$ so that $g * e = e * g = g$.
3) For every $g \in G$, there is a unique element $h \in G$ so that $g * h = h * g = e$. We denote this element by g^{-1}.

Definition 2 (Abelian Groups). *A group so that $gh = hg$ for all of its elements.*

See that transformations of the square do not form an abelian group.

0.2 Fundamental Group

- Paths and Homotopy

We denote by I the unit interval in $\mathbb{R}$.

Definition 3. *A path in a "space" X is a continuous map $\gamma : I \to X$.*

Examples: Some curves in the line, in the plane, in the 3-space, in the circle, in the torus, that cross itself, cover itself.

Definition 4. *A loop is a path that starts and ends at the same points.*

- Drawing examples of homotopy (the linear one, the one in the torus, example in the pinched plane).

Definition 5 (Homotopy). *A (fixed endpoint) homotopy between two paths γ_0, γ_1 in X is a continuous map $H : \boldsymbol{IxI} \to X$ so that $H(*,0) = \gamma_0$, $H(*,1) = \gamma_1$, and $H(0,*)$, $H(1,*)$ are fixed.*

Definition 6 (Discretizing information). *Let γ_0, γ_1 be two loops. We say that $\gamma_0 \simeq \gamma_1$ if there is a homotopy between them. Comment: This is an equivalence relation(to be checked).*

- The loops at a fixed point in the plane to show why are we really discretizing information.

Definition 7 (The group operation). *Let γ_1, γ_2 be two equivalence classes of loops in X. Then we define $[\gamma_1] * [\gamma_2]$ to be the loop that for the first half does what γ_1 does and then for the second half does what γ_2 does.*

Definition 8 (Vague definition). *A neighbourhood of a point is an open-k-disc like sub-ambient(importance of ambient) around that point.*

-Examples: in the real line, in the circle, in the plane, and in the half plane.

Definition 9. *Two spaces X and Y are homeomorphic spaces if there are continuous maps $f : X \to Y$ and $g : Y \to X$ so that $fg = 1_Y$ and $gf = 1_X$.*

0.3 Covering Space

- Example of circle covered by the real line.

Definition 10 (Covering space). *Let $\pi : E \to B$ be a surjective continuous map between two spaces E and B so that for each point $x \in B$, there is a neighbourhood U around x so that $\pi^{-1}(U)$ is a union of non-interacting equivalent spaces V_i to U, which we write as $\coprod_i V_i$ so that $\pi|_{V_i} : V_i \to U$ is an equivalence. Then, we call the triple (π, E, B) a covering space.*

- examples to demonstrate covering spaces.
-Simply connected space

Theorem 0.1. *Let $\pi : E \to B$ be a covering map. Assume that E is simply connected. Then, the deck transformations of the universal cover*

Fact: Let $p : \tilde{X} \to X$ be a covering space map. After choosing initial lift of the starting point of path, then, paths in X lift uniquely to $\tilde{X}$.
Fact: Fixed endpoint homotopies in X are lifted uniquely up to choosing the basepoint's lift.
Theorem: $\pi_1(S^1) = \mathbb{Z}$.

Proof. Consider $\mathbb{R} \to S^1$ via $t \mapsto (cos(2\pi t), sin(2\pi t))$. This is a covering space map. Also, the top space is simply connected. Let ω_n be the loop that loops n times around the circle. Then, pick any other loop around S^1. Take the lift. Observe that it has to be fixed endpoint homotopy above. Push down. Lastly, assume that $[\omega_n] = [\omega_m]$. Homotopy below would lift to fixed endpoint homotopy above. But $n \neq m$, hence we are done. Thus, $\pi_1(S^1) = \mathbb{Z}$. □

A question to answer: The sphere and the torus are not homeomorphic.
Fact: Universal covering space of the torus is the plane.

Definition 11. *Deck transformations of a covering map.*

Theorem 0.2. *Let B be a connected space. Let $\pi : E \to B$ be <u>the</u> universal covering space of B (with B connected). Then, the deck transformations of the universal cover, as a group, give us the fundamental group of B.*

$$Deck(\pi : E \to B) \cong \pi_1(B)$$

Categories and Functors

Examples of Categories: Top, Gp. Example of functor: Fundamental group.
Theorem: Every continuous map $h : D^2 \to D^2$ has a fixed point $x \in D^2$.

Proof. Assume not. This gives us a continuous retraction $D^2 \to \partial D^2$ (note how construction is a little backwards). Note that $\pi_1(D^2) = 0$. Thus, by functoriality of π_1 we are done. □

0.4 Homework

1. Look in the notes and try to state formally what the homotopy lifting property (for covering spaces) means.

2. Describe what you expect the fundamental group of the torus to be and the reasoning (as far as you can) behind your intuition. In particular, I want you to find the universal covering space for the torus.

□