ORMC: BASIC LINEAR ORDINARY DIFFERENTIAL EQUATIONS

OLYMPIAD GROUP 1, WEEK 3

Last time. We have seen that differential equations occur naturally in Physics, often in the form $y''(t) = \text{something}(t, y(t), y'(t))$. For instance,

(1) Gravity: $y''(t) = -g$. Gravitational force applies a constant downward acceleration.
(2) Air drag: $y''(t) = -\alpha y'(t)$. When an object is moving through air, the air particles push it back, with a force proportional to its speed.
(3) Spring: $y''(t) = -\alpha y(t)$. An elastic spring exerts a backwards force proportional to its length.

We considered gravity last time, and briefly introduced the exponential function, which we need for air drag.

Exponentials. We have defined

$$\exp(x) \coloneqq 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \tag{1}$$

and have seen that it satisfies

(1) $\exp(x) = e^x$, where $e \coloneqq \exp(1) \approx 2.718$
(2) $(e^{ax})' = a \cdot e^{ax}$
(3) The solutions to $f' = af$ are precisely $c \cdot e^{ax}$ where c denotes a constant.

Problem 1. (Air drag, a.k.a Damped movement) A small object is sliding on an ice rink, and it's being slowed down only by air drag. Denote by $v(t)$ the object's speed at time t, and let $v_0 = v(0)$. Assume that air drag is proportional to the object's speed, so we have

$$v'(t) = -\alpha v(t),$$

for some constant $\alpha > 0$.

(a) Show that the speed of the object at any given time $t \geqslant 0$ is

$$v(t) = v_0 e^{-\alpha t}.$$

Does the cube ever stop moving in this model?

(b) If the initial position of the object is $y(0) = 0$, what is the position $y(t)$ at time t? (Assume that the object is moving in the direction in which our y increases, so we have $y'(t) = v(t)$.)

(c) What distance will the object have traveled after a very, very long time? How is this possible if the cube never stops moving in this model?

Problem 2. (Springs, a.k.a. Harmonic oscillators)

(a) Explain, informally, why the summations

$$\begin{aligned} \sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots \\ \cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots \end{aligned} \tag{2}$$

should satisfy $\sin' = \cos$ and $\cos' = -\sin$.

(b) Many physics problems (harmonic oscillators) lead to the differential equation

$$y''(t) = -\omega^2 y(t),$$

for some $\omega > 0$. For example, an object attached to an elastic spring will experience a backwards force, proportional to the displacement $y(t)$. Show that the functions

$$\sin(\omega t) \qquad \text{and} \qquad \cos(\omega t)$$

satisfy this differential equation. Can you find any other solutions? *Hint: Linear combinations.*

General problem. Let $f^{(n)}$ denote the n-th derivative of f, and consider differential equations of the form

$$f^{(n)} + a_{n-1}f^{(n-1)} + \cdots + a_1 f' + a_0 f = 0. \tag{3}$$

This is called a *homogeneous* linear ordinary differential equation.

Problem 3. Assume that $f(t) = e^{rt}$ satisfies the differential equation 3. Show that r must be a root of the polynomial

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

called the *characteristic polynomial*.

Problem 4. Check that $f(x) = xe^x$ satisfies the differential equation $f'' - 2f' + f = 0$. Is this of the form in Problem 3?

[*Review of complex numbers*]

$(a+ib)+(c+id) = (a+b)+i(c+d)$
$(a+ib)(c+id) = (ac-bd)+(ad+bc)i$
$\dfrac{1}{a+ib} = \dfrac{a-ib}{a^2+b^2} = \dfrac{a}{a^2+b^2} + i\dfrac{b}{a^2+b^2}$

Complex numbers can be represented on a cartesian plane, by the rule $x + iy \leftrightarrow (x, y)$. This representation makes it easy to add complex numbers visually as vectors; for multiplication the polar representation is more useful:

$$z = x + iy = \sqrt{x^2+y^2} \cdot \left(\frac{x}{x^2+y^2} + i\frac{y}{x^2+y^2} \right) = |z|(\cos\arg z + i \sin\arg z). \tag{4}$$

Definition. Given complex $z = x + iy$, we define its exponential by the same series as in (1):

$$\begin{aligned} e^z = \exp(z) &:= 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots \\ &= 1 + (x+iy) + \frac{(x+iy)^2}{2!} + \cdots + \frac{(x+iy)^n}{n!} + \cdots \end{aligned} \tag{5}$$

This ensures that properties of the exponential such as $e^{a+b} = e^a \cdot e^b$ and $(e^{at})' = ae^{at}$ remain true when a and b are complex (to differentiate a complex function of t, we just differentiate its real and imaginary parts separately).

Problem 5. (Properties of the complex exponential)

(a) Show (informally), using (5) and (2), that if $z = iy$ is a purely imaginary number then

$$e^{iy} = \cos y + i \sin y.$$

In particular, $e^{x+iy} = e^x(\cos y + i\sin y)$. Compare this to the polar representation in (4); which complex numbers can be represented as $e^{\text{something}}$?

(b) Using the relation $e^{ix} \cdot e^{iy} = e^{i(x+y)}$, derive the trigonometric formulas

$$\cos(x+y) = \cos x \ \cos y - \sin x \ \sin y, \qquad \sin(x+y) = \sin x \ \cos y + \cos x \ \sin y.$$

(c) We haven't actually shown that the series in (2) define the same trigonometric functions cos and sin that you know from geometry, but hopefully part (b) makes that claim believable. Assuming this, prove the famous relation

$$\boxed{e^{i\pi} = -1.}$$

Theorem (Hard) Given any set of initial conditions $f(0) = s_0, f'(0) = s_1, \ldots, f^{(n-1)}(0) = s_{n-1}$, there is a unique solution to the differential equation (3). Moreover, this solution can be written as a unique linear combination of fundamental solutions of the form $x^d \cdot e^{rx}$, with r a root of the characteristic polynomial, and d less than the multiplicity of r.

Problem 6. Solve the differential equation $f''(t) - 4f'(t) + 3f(t) = 0$, under the initial conditions $f(0) = 2, f'(0) = 4$.

Problem 7. Solve the differential equation $f''(t) - 2f'(t) + 2f(t) = 0$, under the initial conditions $f(0) = f'(0) = 1$.

Problem 8. Solve the differential equation $f'''(t) + f''(t) = f'(t) + f(t)$, under the initial conditions $f(0) = 2, f'(0) = 1, f''(0) = 0$.

Below is a summary of useful formulas (where c is a constant):

$(f+g)' = f' + g'$	$c' = 0$
$(c \cdot f)' = c \cdot f'$	$x' = 1$
$(f \cdot g) = f' \cdot g + f \cdot g'$	$(x^n)' = nx^{n-1}$
$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$	$(e^{cx})' = ce^{cx}$
$(f \circ g)' = (f' \circ g) \cdot g'$	$\sin' = \cos, \quad \cos' = -\sin$

Homework

Problem 1. (Inhomogeneous equations) Consider the slightly more general differential equation

$$f^{(n)} + a_{n-1}f^{(n-1)} + \cdots + a_1 f' + a_0 f = c,$$

where c is a constant and $a_0 \neq 0$. In class, we have only studied the case $c = 0$. Plug in $f(t) = g(t) + \frac{c}{a_0}$ above, and show how one can use this trick to reduce to the case $c = 0$.

Problem 2. (Damped falling object) Consider the physical example of a falling object, but in which one also takes air drag into account. That is, besides the gravitational acceleration g, we also have an acceleration as in Problem 1. Combined, they give the differential equation

$$y''(t) = -g - \alpha y'(t) \qquad \Longleftrightarrow \qquad v'(t) + \alpha v(t) + g = 0,$$

for some $\alpha > 0$, where $v = y'$ is the object's velocity (positive velocity means upwards movement). Assume that the object has initial velocity of $v(0) = v_0$. Use the trick in Problem 1 to compute $v(t)$ as a function of t. If we let the object fall for a very long time, what will happen to the object's speed $|v(t)|$ (will it increase indefinitely, or will it approach a finite value)?