ORMC: BASIC LINEAR ORDINARY DIFFERENTIAL EQUATIONS

OLYMPIAD GROUP 1, WEEK 3

Last time. We have seen that differential equations occur naturally in Physics, often in the form \( y''(t) = \text{something}(t, y(t), y'(t)) \). For instance,

1. Gravity: \( y''(t) = -g \). Gravitational force applies a constant downward acceleration.
2. Air drag: \( y''(t) = -\alpha y'(t) \). When an object is moving through air, the air particles push it back, with a force proportional to its speed.
3. Spring: \( y''(t) = -\alpha y(t) \). An elastic spring exerts a backwards force proportional to its length.

We considered gravity last time, and briefly introduced the exponential function, which we need for air drag.

Exponentials. We have defined

\[
\exp(x) := 1 + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots
\]

and have seen that it satisfies

1. \( \exp(x) = e^x \), where \( e := \exp(1) \approx 2.718 \)
2. \( (e^{ax})' = a \cdot e^{ax} \)
3. The solutions to \( f' = af \) are precisely \( c \cdot e^{ax} \) where \( c \) denotes a constant.

Problem 1. (Air drag, a.k.a Damped movement) A small object is sliding on an ice rink, and it’s being slowed down only by air drag. Denote by \( v(t) \) the object’s speed at time \( t \), and let \( v_0 = v(0) \). Assume that air drag is proportional to the object’s speed, so we have \( v'(t) = -\alpha v(t) \),

for some constant \( \alpha > 0 \).

(a) Show that the speed of the object at any given time \( t \geq 0 \) is \( v(t) = v_0 e^{-\alpha t} \).

(b) If the initial position of the object is \( y(0) = 0 \), what is the position \( y(t) \) at time \( t \)? (Assume that the object is moving in the direction in which our \( y \) increases, so we have \( y'(t) = v(t) \).)

(c) What distance will the object have traveled after a very, very long time? How is this possible if the cube never stops moving in this model?

Problem 2. (Springs, a.k.a. Harmonic oscillators)

(a) Explain, informally, why the summations

\[
\begin{align*}
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots \\
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots
\end{align*}
\]

should satisfy \( \sin' = \cos \) and \( \cos' = -\sin \).
(b) Many physics problems (harmonic oscillators) lead to the differential equation
\[ y''(t) = -\omega^2 y(t), \]
for some \( \omega > 0 \). For example, an object attached to an elastic spring will experience a backwards force, proportional to the displacement \( y(t) \). Show that the functions
\[ \sin(\omega t) \quad \text{and} \quad \cos(\omega t) \]
satisfy this differential equation. Can you find any other solutions? Hint: Linear combinations.

**General problem.** Let \( f^{(n)} \) denote the \( n \)-th derivative of \( f \), and consider differential equations of the form
\[ f^{(n)} + a_{n-1}f^{(n-1)} + \cdots + a_1 f' + a_0 f = 0. \]  
This is called a **homogeneous** linear ordinary differential equation.

**Problem 3.** Assume that \( f(t) = e^{rt} \) satisfies the differential equation 3. Show that \( r \) must be a root of the polynomial
\[ x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \]
called the **characteristic polynomial**.

**Problem 4.** Check that \( f(x) = xe^x \) satisfies the differential equation \( f'' - 2f' + f = 0 \). Is this of the form in Problem 3?

[Review of complex numbers]

\[
\begin{align*}
(a + ib) + (c + id) &= (a + b) + i(c + d) \\
(a + ib)(c + id) &= (ac - bd) + (ad + bc)i \\
\frac{1}{a + ib} &= \frac{a - ib}{a^2 + b^2} = \frac{a}{a^2 + b^2} + i \frac{b}{a^2 + b^2}
\end{align*}
\]

Complex numbers can be represented on a cartesian plane, by the rule \( x + iy \leftrightarrow (x, y) \). This representation makes it easy to add complex numbers visually as vectors; for multiplication the polar representation is more useful:
\[ z = x + iy = \sqrt{x^2 + y^2} \cdot \left( \frac{x}{\sqrt{x^2 + y^2}} + i \frac{y}{\sqrt{x^2 + y^2}} \right) = |z|(\cos \arg z + i \sin \arg z). \]  

**Definition.** Given complex \( z = x + iy \), we define its exponential by the same series as in (1):
\[ e^z = \exp(z) := 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots \]
\[ = 1 + (x + iy) + \frac{(x + iy)^2}{2!} + \cdots + \frac{(x + iy)^n}{n!} + \cdots \]  
This ensures that properties of the exponential such as \( e^{a+b} = e^a \cdot e^b \) and \( (e^{at})' = ae^{at} \) remain true when \( a \) and \( b \) are complex (to differentiate a complex function of \( t \), we just differentiate its real and imaginary parts separately).

**Problem 5.** (Properties of the complex exponential)

(a) Show (informally), using (5) and (2), that if \( z = iy \) is a purely imaginary number then
\[ e^{iy} = \cos y + i \sin y. \]
In particular, \( e^{x+iy} = e^x(\cos y + i \sin y) \). Compare this to the polar representation in (4); which complex numbers can be represented as \( e^{\text{something}} \)?
(b) Using the relation $e^{ix} \cdot e^{iy} = e^{i(x+y)}$, derive the trigonometric formulas
\[
\cos(x + y) = \cos x \cos y - \sin x \sin y, \quad \sin(x + y) = \sin x \cos y + \cos x \sin y.
\]

(c) We haven't actually shown that the series in (2) define the same trigonometric functions $\cos$ and $\sin$ that you know from geometry, but hopefully part (b) makes that claim believable. Assuming this, prove the famous relation
\[
e^{i\pi} = -1.
\]

**Theorem** (Hard) Given any set of initial conditions $f(0) = s_0, f'(0) = s_1, \ldots, f^{(n-1)}(0) = s_{n-1},$ there is a unique solution to the differential equation (3). Moreover, this solution can be written as a unique linear combination of fundamental solutions of the form $x^d \cdot e^{rx},$ with $r$ a root of the characteristic polynomial, and $d$ less than the multiplicity of $r$.

**Problem 6.** Solve the differential equation $f''(t) - 4f'(t) + 3f(t) = 0,$ under the initial conditions $f(0) = 2, f'(0) = 4.$

**Problem 7.** Solve the differential equation $f''(t) - 2f'(t) + 2f(t) = 0,$ under the initial conditions $f(0) = f'(0) = 1.$

**Problem 8.** Solve the differential equation $f'''(t) + f''(t) = f'(t) + f(t),$ under the initial conditions $f(0) = 2, f'(0) = 1, f''(0) = 0.$

Below is a summary of useful formulas (where $c$ is a constant):

<table>
<thead>
<tr>
<th>$(f + g)' = f' + g'$</th>
<th>$c' = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(c \cdot f)' = c \cdot f'$</td>
<td>$x' = 1$</td>
</tr>
<tr>
<td>$(f \cdot g)' = f' \cdot g + f \cdot g'$</td>
<td>$(x^n)' = nx^{n-1}$</td>
</tr>
<tr>
<td>$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$</td>
<td>$(e^{cx})' = ce^{cx}$</td>
</tr>
<tr>
<td>$(f \circ g)' = (f' \circ g) \cdot g'$</td>
<td>$\sin' = \cos, \quad \cos' = -\sin$</td>
</tr>
</tbody>
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**Homework**

**Problem 1.** (Inhomogeneous equations) Consider the slightly more general differential equation
\[
f^{(n)} + a_{n-1}f^{(n-1)} + \cdots + a_1 f' + a_0 f = c,
\]
where $c$ is a constant and $a_0 \neq 0.$ In class, we have only studied the case $c = 0.$ Plug in $f(t) = g(t) + \frac{e^{\alpha t}}{a_0}$ above, and show how one can use this trick to reduce to the case $c = 0.$

**Problem 2.** (Damped falling object) Consider the physical example of a falling object, but in which one also takes air drag into account. This is, besides the gravitational acceleration $g,$ we also have an acceleration as in Problem 1. Combined, they give the differential equation
\[
y''(t) = -g - \alpha y'(t) \quad \iff \quad v'(t) + \alpha v(t) + g = 0,
\]
for some $\alpha > 0,$ where $v = y'$ is the object’s velocity (positive velocity means upwards movement). Assume that the object has initial velocity of $v(0) = v_0.$ Use the trick in Problem 1 to compute $v(t)$ as a function of $t.$ If we let the object fall for a very long time, what will happen to the object’s speed $|v(t)|$ (will it increase indefinitely, or will it approach a finite value)?