Week 4: Combinatorics continued

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1 From Before

Problem 1 (Week 1 Problem 2).

a) Show that
\[ \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n. \]

b) Show that
\[ \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots + (-1)^n \binom{n}{n} = 0. \]

Problem 2 (Week 2 Problem 6).
Show that if \( p \) is prime and \( 1 \leq k < p \), then \( p \mid \binom{p}{k} \).

Problem 3 (Week 2 Problem 7).
Let \( ABCD \) be a cyclic quadrilateral, and let \( T \) be the intersection of lines \( AB \) and \( CD \). Assume \( A \) lies on the segment \( TB \) and \( D \) lies on the segment \( TC \). Show that \( TA \cdot TB = TC \cdot TD \).

Problem 4 (Week 2 Problem 8).
Let \( BB_1 \) and \( CC_1 \) be altitudes in a triangle \( \triangle ABC \). Show that the tangent line at \( A \) to the circumcircle of \( \triangle ABC \) is parallel to \( B_1C_1 \).

2 New Problems

Problem 1.
Toys R Us has recently introduced a new revolutionary type of toy – a wire cube with a colored sphere at each corner. The spheres can be one of 8 colors, and each cube has to contains all 8 possible colors. How many different cubes can Toys R Us produce?

Problem 2.
Count the number of 5-digit numbers which contains exactly the digits 1,2,3,4,5 and the even digits are not adjacent to each other.

Problem 3.
Prove that for any integer \( a, b \) and prime \( p \) one has
\[ p \mid (a + b)^p - a^p - b^p. \]