Week 4: Combinatorics continued

Konstantin Miagkov, Nikita

1 From Before

Problem 1 (Week 1 Problem 2).
a) Show that

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n.$$

b) Show that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots + (-1)^n \binom{n}{n} = 0.$$

Problem 2 (Week 2 Problem 6).
Show that if p is prime and $1 \leq k < p$, then $p \mid \binom{p}{k}$.

Problem 3 (Week 2 Problem 7).
Let $ABCD$ be a cyclic quadrilateral, and let T be the intersection of lines AB and CD. Assume A lies on the segment TB and D lies on the segment TC. Show that $TA \cdot TB = TC \cdot TD$.

Problem 4 (Week 2 Problem 8).
Let BB_1 and CC_1 be altitudes in a triangle $\triangle ABC$. Show that the tangent line at A to the circumcircle of $\triangle ABC$ is parallel to B_1C_1.

2 New Problems

Problem 1.
Toys R Us has recently introduced a new revolutionary type of toy – a wire cube with a colored sphere at each corner. The spheres can be one of 8 colors, and each cube has to contains all 8 possible colors. How many different cubes can Toys R Us produce?

Problem 2.
Count the number of 5-digit numbers which contains exactly the digits 1,2,3,4,5 and the even digits are not adjacent to each other.

Problem 3.
Prove that for any integer a, b and prime p one has

$$p \mid (a + b)^p - a^p - b^p.$$