2021 USAJMO Review

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The 2021 USAJMO problems can be found here `https://artofproblemsolving.com/community/c1986133_2021_usajmo`, and the 2021 USAMO problems can be found here `https://artofproblemsolving.com/community/c1986140_2021_usamo`. The two contests shared 3 problems. I aim to cover all USAJMO problems plus USAMO/3 and explain motivations for their solutions. There is no overarching theme except mathematical insight at the Olympiad level. Note that there would be no expectation to solve these in 2 hours.

1 USAJMO 1

Problem 1. Let $\mathbb{N}$ denote the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all positive integers a and b,

$$f(a^2 + b^2) = f(a)f(b)\,[1] \text{ and } f(a^2) = f(a)^2\,[2].$$

(I have labeled the first condition [1] and the second condition [2].)

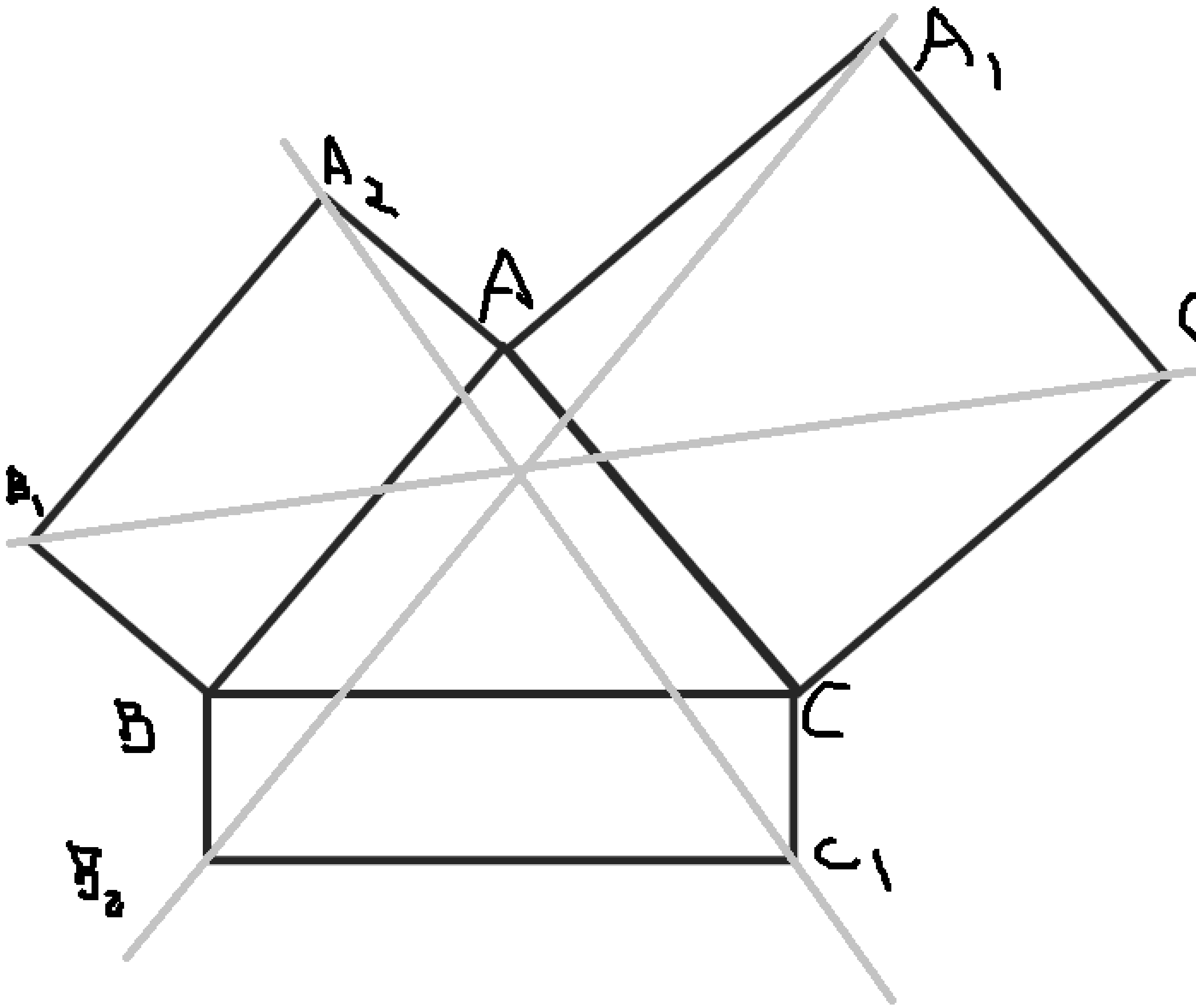
(You can skip the following if you know what a functional equation is). Functional equation problems are a mixed bag. We start out considering all functions $\mathbb{N} \to \mathbb{N}$, of which there are a huge number, and then there is a class of conditions, typically infinite. We should try to understand how different function values, e.g. $f(1), f(2), f(3)$ are related, try to simplify the given conditions, and place restrictions on a valid function f. Try to guess the set of answers or at least one function that works. Next, set a, b equal to small positive integers in [1] and [2].

2 USAJMO 2

Problem 2. Rectangles BCC_1B_2, CAA_1C_2, and ABB_1C_2 are erected outside an acute triangle ABC. Suppose that

$$\angle BC_1C + \angle CA_1A + \angle AB_1B = 180°.$$

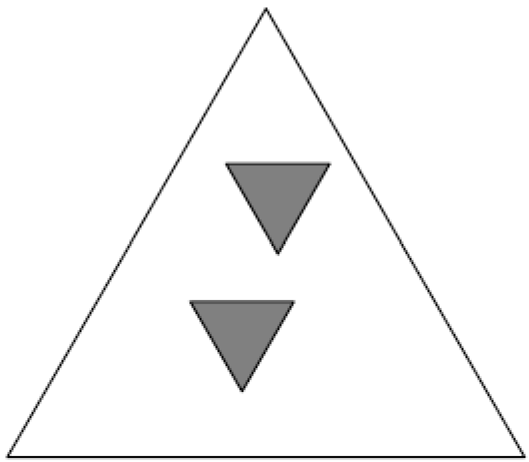
Prove that lines B_1C_2, C_1A_2, and A_1B_2 are concurrent.



3 USAJMO 3

Problem 3. An equilateral triangle Δ of side length $L > 0$ is given. Suppose that n equilateral triangles with side length 1 and with non-overlapping interiors are drawn inside Δ such that each unit equilateral triangle has sides parallel to Δ, but with opposite orientation. An example with $n = 2$ is shown below. Prove that

$$n \leq \frac{2}{3}L^2.$$



You should see that $\lfloor L \rfloor^2$ triangles whose orientation is allowed to be "the same" or "the opposite" as Δ can fit inside Δ, and that a fraction almost 1/2 of these have the correct orientation. What seems to be a dense tiling of the plane of equilateral triangles of the same orientation?

4 USAJMO 4

Problem 4. Carina has three pins, labeled A, B, and C, respectively, located at the origin of the coordinate plane. In a move, Carina may move a pin to an adjacent lattice point at distance 1 away. What is the least number of moves that Carina can make in order for triangle ABC to have area 2021? (A lattice point is a point (x; y) in the coordinate plane where x and y are both integers, not necessarily positive.)

Hint: What is the largest area of a triangle that you can think of after n moves? E.g. $n = 100$ or $n = 101$.

Note: The area of any triangle in the plane whose vertices are lattice points is an integer of half-integer. To be fancy, the area is in $(1/2)\mathbb{Z}$. Also, using the fact that any simple polygon in the plane has a triangulation, the area of any polygon whose vertices are lattice points is in $(1/2)\mathbb{Z}$.

5 USAJMO 5

Problem 5. A finite set S of positive integers has the property that, for each $s \in S$, and each positive integer divisor d of s, there exists a unique element $t \in S$ satisfying $gcd(s,t) = d$. (The elements s and t could be equal.) Given this information, find all possible values for the number of elements of S.

The conditions impose a strong condition on the size of S and an element in s. What is it? Next, consider the prime factorization of s.

6 USAJMO 6

Problem 6. Let $n \geq 4$ be an integer. Find all positive real solutions to the following system of 2n equations:

$$
\begin{aligned}
a_1 &= \frac{1}{a_{2n}} + \frac{1}{a_2}, & a_2 &= a_1 + a_3, \\
a_3 &= \frac{1}{a_2} + \frac{1}{a_4}, & a_4 &= a_3 + a_5, \\
a_5 &= \frac{1}{a_4} + \frac{1}{a_6}, & a_6 &= a_5 + a_7, \\
&\vdots & &\vdots \\
a_{2n-1} &= \frac{1}{a_{2n-2}} + \frac{1}{a_{2n}}, & a_{2n} &= a_{2n-1} + a_1.
\end{aligned}
$$

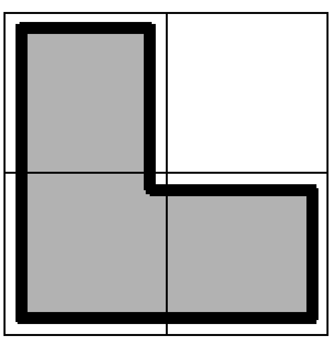
On the math olympiads, symmetric inqequalities in 3 real variables have been declining in frequency. Instead, there are things like n variables and something breaking the symmetry. In this case, it is n variables, very slight asymmetry, and in fact all equalities! (Perhaps comparing this problem to symmetric inequalities is a stretch.) This particular problem does not do justice to the variety of methods involved in these algebra problems, which we will have to defer.

7 USAMO 3

Problem 7. Let $n \geq 2$ be an integer. An $n \times n$ board is initially empty. Each minute, you may perform one of three moves:

1. If there is an L-shaped tromino region of three cells without stones on the board (see figure; rotations not allowed), you may place a stone in each of those cells.
2. If all cells in a column have a stone, you may remove all stones from that column.
3. If all cells in a row have a stone, you may remove all stones from that row.

For which n is it possible that, after some non-zero number of moves, the board has no stones?



From small examples, you should figure out that $n = 2$ does not work, $n = 3$ works, and $n = 4$ does not seem to work, but the last is hard to prove to start. Let us consider a different problem where we have a strictly greater set of allowed moves.

Instead of each square having 0/1 stones, instead each square has an integer counter that starts at 0 and can go positive/negative. We have 3 stronger moves possible. (1) In any L-shaped tromino, we can increment all 3 cells. (without any limitation!) (2) In any column, we can decrement all cells (3) In any row, we can decrement all rows.

(4) However, we cannot affect the top row or rightmost column. Now, there are a finite number of move-types and it suffices to keep track of the counts of moves taken. This results in a a positive integer combination of the moves. Now, **the stronger claim is that** any nonzero integer combination of the moves is nonzero.

Now there is an advanced method, which is interpreting the problem as a generating function in two variables.

8 In case you are bored and know all solutions

Exercise 1. In USAMO 3, instead suppose that we allow rotations in the L-tromino in the first move, and the rest of the problem is the same. Now what values of n are possible?

Exercise 2. What is the linear algebra statement that is suggested by USAMO 3? Note that your statement does not need to be implied by USAMO 3 because your statement should be a stronger statement. Note that the linear algebra problem will be over $\mathbb{Q}$. Check the dimensions of this problem.

Problem 3. (Very hard) Let P be a cyclic and convex polygon in the coordinate plane whose vertices are all lattice points. Suppose that n is an odd integer such that for each side length d of P, d^2 is a multiple of n. Let S be the area of P. Prove that $2S$ is a multiple of n.