# LATTICE FUN

OLGA RADKO MATH CIRCLE
ADVANCED 2
APRIL 18, 2021

*Orchard in full bloom*
*Apples litter the damp ground*
*Sunrise come and gone*

## 1. Introduction

Everybody loves whole numbers. Seriously, what's better than a nice integer? This is why number theory often deals with integer solutions. For example, Fermat's Last Theorem says that there are no non-trivial integer solutions to $a^n + b^n = c^n$ for $n > 2$. Consider the case where $n = 3$. The set of solutions to $x^3 + y^3 = z^3$ is a 2-dimensional surface in $\mathbb{R}^3$. Fermat's Last Theorem is a statement about this surface's (and other surfaces' for different values of n) avoidance of the integer points in $\mathbb{R}^3$. Take your favorite number theoretic result and turn it geometric! In this handout we will develop some tools to solve problems of this form (not FLT), granted they satisfy some extra conditions. We will then apply this to a neat little problem regarding orchards.

**Definition 1.** The **integer lattice** $\mathbb{Z}^n$ in $\mathbb{R}^n$ is the set of points with integer coordinates. Written as a set, $\mathbb{Z}^n = \{(a_1, a_2, \ldots, a_n) | a_i \in \mathbb{Z}\}$. We call each point in the lattice a **lattice point**.

The most common example is the integer lattice $\mathbb{Z}^2$ in $\mathbb{R}^2$. This is the set of points in the plane with integer coordinates.

**Problem 1.** Draw the integer lattice $\mathbb{Z}^2$.

We say that a set of vectors $\{v_1, v_2, \ldots, v_n\}$ **generates** $\mathbb{Z}^n$ if every lattice point can be written uniquely as $a_1 v_1 + \cdots + a_n v_n$ where the coefficients $a_i$ are integers.

**Problem 2.** Find a set of vectors which generate the integer lattice $\mathbb{Z}^n$. Make sure your set has as few vectors as possible, this will ensure uniqueness.

**Problem 3.** Which of the following generate $\mathbb{Z}^2$

(1) $\{(1, 2), (2, 1)\}$
(2) $\{(1, 0), (0, 2)\}$
(3) $\{(1, 1), (1, 0), (0, 1)\}$

**Definition 2.** We call the parallelepiped spanned by a generating set of $\mathbb{Z}^n$ the **fundamental region** of the lattice.

While the region itself might depend on the choice of generating set, the volume does not.

**Problem 4.** Draw a picture of the fundamental region of $\mathbb{Z}^3$. What is its volume?

## 2. Minkowski's Theorem

Many problems in number theory boil down to finding integer solutions to a problem which can be extended over $\mathbb{R}^n$. Our goal in this section is to develop some tools to prove existence of integer solutions inside a set of general (real-valued) solutions.

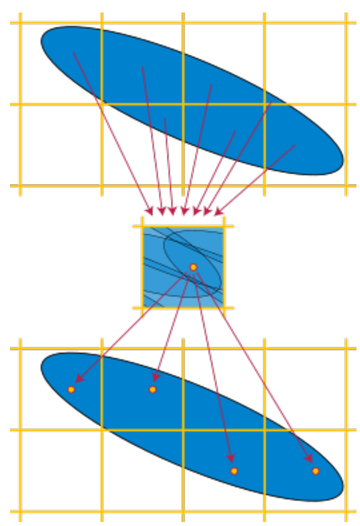
**Theorem 1.** (Blichfeldt) Let $X \subset \mathbb{R}^n$ be a set of finite volume. If $Vol(X) > 1$, then $X$ contains two distinct points which differ by an element of $\mathbb{Z}^n$. That is, there exist distinct $x, y \in X$ such that $x - y \in \mathbb{Z}^n$.

Blichfeldt's theorem says that if a set $X$ is large enough (volume-wise), then it can be translated in the plane to contain two integer points.

**Problem 5.** Draw a region in the plane with volume greater than 1 which does not contain any lattice points. Find two points in the region which differ by an integer vector.

Let us first prove Blichfeldt's theorem for $\mathbb{Z}^2 \subset \mathbb{R}^2$.

**Problem 6.** The following picture gives the idea for the proof of Blichfeldt's theorem. Explain the picture and complete the proof.



**Problem 7.** Does your proof of Blichfeldt's theorem for $\mathbb{Z}^2$ extend to a proof of Blichfeldt's for $\mathbb{Z}^n$?

**Problem 8.** Let $X \subset \mathbb{R}^n$ be a set of volume $k$. What is the maximum number of integer points we can guarantee to be in X after a possible translation?

We now know that with a large enough region and the freedom to move it around in the plane, we can guarantee multiple lattice points in the region. However, our problems are often sensitive to translations and depend on a rigid region. The following theorem removes translations but introduces additional structure on the region we are concerned with. Specifically, we consider convex sets which are symmetric with respect to the origin.

**Definition 3.** A set $X \subset \mathbb{R}^n$ is **convex** if the line segment connecting any two points in $X$ lies entirely in X.

**Problem 9.** (1) Draw a convex region in the plane, and indicate on your drawing what the convex condition looks like.
(2) Draw a region which is not convex and prove that it is not convex.

**Problem 10.** Prove that if $X$ is convex, then the midpoint of any two points in $X$ is also in $X$.

**Definition 4.** A set $X \subset \mathbb{R}^n$ is **symmetric with respect to the origin** if for all $x \in X$, $-x$ is also in $X$.

**Problem 11.** (1) Draw an example of a region in the plane which is symmetric with respect to the origin.
(2) Draw an example of a region in the plane which is not symmetric with respect to the origin.

We are now ready to state Minkowski's theorem.

**Theorem 2.** (Minkowski) Every convex set in $\mathbb{R}^n$ which is symmetric with respect to the origin and which has volume greater than $2^n$ contains a non-zero integer point.

Minkowski's theorem tells us that for a large enough and well-structured set, we are guaranteed a non-trivial lattice point.

**Problem 12.** Let $K$ be our set satisfying the conditions of Minkowski's theorem. Shrink each coordinate of $\mathbb{R}^n$ by a factor of 2 to get a new set, call it $K' := \frac{1}{2}K$.
(1) What is the volume of $K'$ in relation to that of $K$?
(2) Prove that the sum of any two points in $K'$ sits in $K$. (Hint: Convexity might be useful)
(3) Apply Blichfeldt's theorem to $K'$ to prove Minkowski's theorem. (Hint: Symmetry with respect to the origin might be useful)

## 3. Polya's Orchard Problem

*Orchard in full bloom*
*Apples litter the damp ground*
*Sunrise come and gone*

It's spring time; the birds are singing and the flowers are blooming. What a time. Picture this: You are standing in the center of a circular orchard of integer radius R. The trees were planted at the integer lattice points and have each grown to have the same radius r. If the radius is small enough, you will have a clear line-of-sight through the orchard in some direction. If the radius is too large, there is no line-of-sight through the orchard no matter the direction. The following figure is an example, feel free to color it in with whatever type of trees you would like.

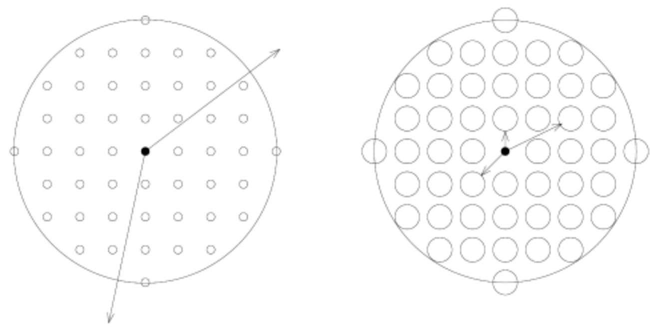


Figure 1. Two orchards of radius 4.

**Problem 13.** Show that if $r < \frac{1}{\sqrt{R^2+1}}$, then there is a clear line-of-sight. (Hint: Take a look at the ray through the point $(R, 1)$ and calculate the distance from the closest integer points to the ray)

**Problem 14.** Prove that if $r > \frac{1}{R}$, then there is no line-of-sight through the orchard. If you'd like, you can use the following steps:

(1) Show that if $r \geq 1$, then there is no line-of-sight.
(2) Now suppose $r < 1$ and $r > \frac{1}{R}$. Then $R \geq 2$. Choose a potential line-of-sight, say it passes through a point P on the circle. Thicken this line-of-sight equally on both sides into a rectangle of width $2r$ tangent to $P$ and $-P$. From here, use Minkowski's to get a contradiction. (Don't forget to rule out any lattice points that sit outside the orchard but inside the rectangle.)

Any interest in counting the number of trees in the orchard? If so, google "the Gauss circle problem". If orchards are not your slice of fruit, maybe rational approximations are your cup of t$ea$.

**Problem 15. (Challenge)** Prove that there exists a rational approximation of $\sqrt{3}$ within $10^{-3}$ with denominator at most 501. Come up with an upper bound for the smallest denominator of a $\epsilon$-close rational approximation of any irrational number $\alpha > 0$. Your bound can have some dependence on $\alpha$ and should get smaller as $\alpha$ gets larger.
(Hint: Consider a line through an orchard with slope $\sqrt{3}$)